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2. *Do not* try to typewrite any complex formulas. Write them.
3. Keep a copy, and send the editors two copies, if you can.
4. *Do not* underline any symbols or any formulas.
5. Underline theorems with blue pencil (avoid ink).
6. Follow our recent styles in abbreviations, footnotes, etc.
7. Write carefully the (often misunderstood) capitals C K P S V W X Z.
8. Write ϵ, not ε. Write very carefully $\gamma\ \eta\ \varepsilon\ \lambda\ \nu\ \tau\ \upsilon\ \chi\ \omega$.
9. Among Greek capitals, use only $\Gamma\ \Delta\ \Theta\ \Lambda\ \Xi\ \Pi\ \Sigma\ \Phi\ \Psi\ \Omega$.
10. Punctuate carefully, especially in formulas; thus: $1, 2, \cdots, n$.
11. Use the solidus (/) to avoid fractions in solid lines.
12. Use fractional exponents to avoid root signs everywhere.
13. Use extra symbols to avoid complicated exponents.
14. In typewritten formulas, 1 means "one"; to indicate "ell" in formulas, backspace and overprint /; thus: l. Similarly, 0 means "zero"; to indicate "cap O," backspace and overprint period; thus: O.
15. Avoid a dash over a letter, except for those shown below.
16. Some samples of unusual types available on monotype machines follow. A more complete list of all such types will be sent on request.

Light Face Greek—$\alpha\ \beta\ \gamma\ \cdots$ (all) A B $\Gamma\ \cdots$ (all).

★ Light Greek Superiors—$^{\Delta}$ and $^{\alpha\ \beta\ \gamma\ \cdots}$ (all except $^{\iota}$ and ^{o}).

★ Light Greek Inferiors—$_{\Delta\ \Lambda\ Z\ \Omega}$ and $_{\alpha\ \beta\ \gamma\ \cdots}$ (all except $_{\iota}$ and $_{o}$).

* Boldface Greek—$\boldsymbol{\alpha\ \beta\ \delta\ \varepsilon\ \zeta\ \eta\ \theta\ \mu\ \nu\ \xi\ \pi\ \rho\ \sigma\ \omega}$ and $\boldsymbol{\Omega}$.

* Lightface German—$\mathfrak{a\ b\ c\ d\ p\ q}\ \mathfrak{A\ B\ C\ D\ E\ F\ G\ H\ J\ L\ M\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$.

* Boldface German—$\mathfrak{b}\ \mathfrak{A\ B\ T}$.

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★ Dashed Italics—$\overline{A}\ \overline{a}\ \overline{B}\ \overline{b}\ \overline{C}\ \overline{c}\ \overline{E}\ \overline{e}\ \overline{F}\ \overline{f}\ \overline{G}\ \overline{g}\ \overline{H}\ \overline{I}\ \overline{i}\ \overline{K}\ \overline{k}\ \overline{M}\ \overline{m}\ \overline{N}\ \overline{n}\ \overline{P}\ \overline{p}\ \overline{q}\ \overline{R}\ \overline{r}\ \overline{s}\ \overline{t}\ \overline{u}\ \overline{v}\ \overline{X}\ \overline{x}\ \overline{Y}\ \overline{y}\ \overline{Z}\ \overline{z}$.

★ Tilda Italics—$\tilde{A}\ \tilde{a}\ \tilde{e}\ \tilde{N}\ \tilde{n}\ \tilde{O}\ \tilde{r}\ \tilde{u}\ \tilde{y}$.

★ Tilda Greek—$\tilde{\alpha}\ \tilde{\epsilon}\ \tilde{\eta}\ \tilde{\omega}\ \tilde{\omega}$.

★ Dashed Greek—$\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}\ \bar{\delta}\ \bar{\eta}\ \bar{\theta}\ \bar{\mu}\ \bar{\nu}\ \bar{\rho}\ \bar{\omega}\ \overline{\Gamma}$.

★ Dotted Italic—$\dot{a}\ \ddot{a}\ \dot{e}\ \ddot{e}\ \ddot{g}\ \ddot{\imath}\ \dot{m}\ \dot{n}\ \dot{q}\ \dot{r}\ \ddot{r}\ \dot{u}\ \ddot{u}\ \dot{x}\ \ddot{x}\ \dot{y}\ \ddot{y}\ \dot{z}\ \ddot{z}$.

★ Dotted Greek—$\dot{\eta}\ \ddot{\eta}\ \dot{\theta}\ \ddot{\theta}\ \dot{\xi}\ \dot{\psi}\ \ddot{\psi}\ \ddot{\omega}$ (single dotted $\zeta\ \phi\ \delta\ \beta\ \gamma$; double dotted γ readily available).

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FIFTY-FIFTH REGULAR MEETING OF THE SAN FRANCISCO SECTION

The fifty-fifth regular meeting of the San Francisco Section of the Society was held at Reed College on Saturday, June 2, 1928. In the absence of Chairman Bell, Professor E. E. DeCov acted as temporary chairman. The total attendance was 26, including the following 19 members of the Society:

J. R. Ballantine, Bernstein, Butler, J. W. Campbell, A. F. Carpenter, D. R. Davis, Donahue, Gavett, Griffin, Hanawalt, Jerbert, Kent, Langford, McAlister, W. E. Milne, Moritz, Mullemeister, Neikirk, Smail.

The Secretary announced that, by invitation of the Program Committee, Professor E. T. Bell will address the Section at its October meeting on the topic *The theory of algebraic numbers in the light of Kronecker's program.*

On motion of Professor Moritz, it was resolved: *That the Section express its sense of loss in the death on May 29 of Professor R. E. Allardice, one of the organizers, a former Chairman, and a contributor to the program of the Section.*

The Secretary announced that Chairman Bell appointed the following committee to nominate officers at the October meeting of the Section: Bernstein, Griffin, E. R. Hedrick.

Professor Kent invited the Section to hold its next Summer meeting at the Oregon State Agricultural College. The invitation was accepted.

Following the meeting, the members and their friends were greeted by President Colman of Reed College at a luncheon given them by the College. A drive up the beautiful Columbia River Highway followed the luncheon.

Titles and abstracts of the papers presented at the meeting follow below. The papers by Ayres, Neikirk, Roberts, and Whyburn were read by title. Miss Haycraft and Miss Nelson were introduced by Professor Griffin. Mrs. Williamson was introduced by Professor Carpenter.

1. Professor B. A. Bernstein: *Irredundant sets of postulates for the logic of propositions.*

Alonzo Church has defined a set of postulates to be *irredundant* if the postulates are independent and no postulate can be "weakened" with respect to the set (Transactions of this Society, vol. 27, p. 318). In the present paper the author gives a number of irredundant sets of postulates for the logic of propositions (see Transactions of this Society, vol. 28, p. 472). The sets are free from the objections usually to be raised against a set of postulates made irredundant by Church's "mechanical" method.

2. Professor J. W. Campbell: *Catenary problems in mechanics.*

The solutions of catenary problems in mechanics are greatly facilitated when in addition to the functions sinh x, cosh x, and tanh x there are available tables of values of $(\sinh x)/x$, $(\cosh x-1))/x$, and $(\cosh x)/x$. Tables of these functions from $x=0.000$ to $x=3.000$ have been compiled and tabulated by the writer and their use in connection with various types of catenary problems shown.

3. Professor A. F. Carpenter: *Tetrads of ruled surfaces.*

In this paper the projective differential properties of four ruled surfaces whose line elements are in correspondence in sets of four, are determined by means of the invariants and covariants of a system of eight ordinary linear homogeneous differential equations of the first order in eight dependent variables, to which are adjoined four linear homogeneous relations between these variables.

4. Professor F. L. Griffin: *Points of minimum travel for a distributed population,* II.

In this paper Professor Griffin discusses further cases of the problem considered at the Vancouver meeting of the Section.

5. Professor F. L. Griffin: *Note on an investment problem involving mean values and probability.*

This note calls attention to an interesting elementary illustration of mean values and probability which occurs in connection with long-term leases containing optional building clauses and accompanying adjustment of rentals, when it is a question of purchasing the lessor's rights at a price to yield a specified rate of interest.

6. Miss Alace Haycraft: *Concerning geodesics on certain surfaces of revolution.*

Miss Haycraft studies in some detail the geodesics on the circular paraboloid and hyperboloid of one sheet, and she contrasts these geodesics with corresponding curves on the cylinder, cone, and sphere.

7. Dr. A. R. Jerbert: *Quadruples of space curves.*

A system of four linear, homogeneous, differential equations of the first order is interpreted as defining, to within a projectivity, four space curves. By virtue of their expression in terms of a common parameter, these curves are in one-to-one point correspondence. If corresponding points are joined by straight lines, a tetrahedron is formed. The study of the four associated curves is accordingly equivalent to the study of a one-parameter family of tetrahedrons. The latter point of view derives some interest from the fact that the tetrahedron configuration occurs frequently in projective differential geometry.

8. Professor W. E. Milne: *The behavior of a boundary value problem as the interval becomes infinite.*

The investigation applies to the linear differential equation $d^2u/dx^2 + G(x, \lambda) = 0$ with linear homogeneous boundary conditions at the ends of an interval $a \leqq x \leqq b$. The object is to determine the behavior of the characteristic numbers, characteristic solutions, and oscillation properties as a and b approach $-\infty$ and $+\infty$ respectively. It is assumed that $G(x, \lambda)$ increases from $-\infty$ to $+\infty$ as λ increases from $-\infty$ to $+\infty$, and also that as x becomes positively or negatively infinite, $G(x, \lambda)$ becomes negatively infinite. Then as the interval becomes infinite, the characteristic numbers approach a set of critical values $\lambda_0, \lambda_1, \lambda_2, \cdots$, which have a limit point at plus infinity only, and which are entirely independent of the boundary conditions. The corresponding characteristic solutions approach critical solutions $u_0, u_1, u_2, \cdots$, which are independent of the boundary conditions. The solution, u, vanishes exactly n times in the infinite interval. Furthermore the integrals from minus infinity to plus infinity of $(du/dx)^2$ and of $G(x, \lambda)u^2$ exist if u is any critical solution, but do not exist for any other solution of the differential equation.

9. Professor R. E. Moritz: *An elementary derivation of the probability function.*

The view-point in the derivation is purely geometric. The procedure is the gradual deformation of the frequency polygon corresponding to binomial distribution $(p+q)^n$. By a proper choice of coordinates and repeated application of Stirling's formula it is shown that as n is indefinitely increased the frequency polygon goes over into the curve $y = \exp(-\pi x^2)$.

10. Professor R. E. Moritz: *A new theory of depreciation.* Preliminary report.

All known methods for the determination of the depreciation of physical assets proceed on the assumption that the life time of the asset is known. In this paper the writer assumes that the repair function is known and attempts to determine the life of the asset so as to make the annual charge to production a minimum.

11. Professor L. I. Neikirk: *A class of totally discontinuous functions.* Second paper.

This paper contains a generalization of the results of the first paper presented to the Section last Spring. The problem is to construct a function $u(x)$ with the following properties: it is equal to some one of the functions $f(x), f_1(x), f_2(x), \cdots, f_r(x)$ at every point of the interval $(0, 1)$, so that if $u(a_1)=f(a_1)$ and $u(a_2)=f(a_2)$, where $0\leqq a_1<a_2\leqq 1$, then at some number, a', where $a_1<a'<a_2$, $u(a')=f_i(a')$, $1\leqq i\leqq r$; and if $u(b_1)=f_i(b_1)$ and $u(b_2)=f_i(b_2)$, where $0\leqq b_1<b_2\leqq 1$, then there is a b', $b_1<b'<b_2$, at which $u(b')=f(b')$; and also if $u(c_1)=f_2(c_1)$ and $u(c_2)=f_s(c_2)$, where $0\leqq c_1<c_2\leqq 1$, then there are numbers c_s', $s=1, 2, \cdots, i-1, i+1, \cdots, r$ and $c_1<c_s'<c_2$ such that $u(c_s')=f_s(c_s')$. The function $u(x)$ is constructed and is defined explicitly.

12. Miss Mildred Nelson: *A further generalization of the circular and hyperbolic functions of sectorial areas.*

The generalization of circular and hyperbolic functions presented by Miss Pennock at the Vancouver meeting of the Section is extended by Miss Nelson. She substitutes an ellipse for the circle and an hyperbola of unequal semi-axes for the usual rectangular hyperbola; also she considers sectional areas from points asymmetrically located as well as from points on the transverse axis.

13. Mr. J. H. Roberts and Mr. J. L. Dorroh: *On a problem by G. T. Whyburn.*

In his recent paper *Concerning irreducible cuttings of continua*, G. T. Whyburn gives an example of a continuum M which contains no indecomposable continuum, but which contains two points A and B, such that no cutting of M between A and B is irreducible. He proves that if M is an indecomposable continuum, then *every* two points A and B of M are such that no irreducible cutting of M between A and B exists. He raises the following questions: (1) If a continuum M is such that no subset of M is an irreducible cutting of M between any two points A and B of M, then is M necessarily indecomposable? (2) If every cutting of a continuum M is reducible, is M necessarily indecomposable? In the present paper it is shown by an example that the answer to both of these questions is in the negative. Also, it is proved that a necessary and sufficient condition that every subcontinuum of a continuum M be indecomposable is that no subcontinuum of M contain an irreducible cutting of itself.

14. Mr. J. H. Roberts: *Concerning closed sets which contain no domain in n-dimensional euclidean space.*

The author has recently shown that if M denotes a plane continuum containing no domain, then there exists a continuous one-to-one transformation of the plane into itself such that in the transformed plane there exists a rectangular coordinate system with the property that if l denotes

any line parallel to one of the coordinate axes, then the set of points common to l and M is either vacuous or totally disconnected. This result is included in the following more general theorem, proved in the present paper: If S is an n-dimensional euclidean space, and M is a closed subset of S containing no n-dimensional domain, then there exists a continuous one-to-one transformation of S into itself, such that in the transformed space every straight line contains at most a totally disconnected subset of M.

15. Professor L. L. Smail: *On the effect of the operation of summability on the interval of oscillation of an infinite sequence.* Preliminary report.

Practically all previous studies of the summability of divergent series have been concerned with the unique limit of an auxiliary sequence associated with the given series. This paper treats of the relation between the upper and lower limits of the auxiliary sequence whose unique limit, when it exists, gives the usual generalized sum of the given sequence. This relation is investigated for the general methods of summability, and also for the Cesàro method when the order of summability is increased.

16. Professor G. T. Whyburn: *Concerning cuttings of continua.*

In this paper it is shown that in a locally compact metric and separable space: (1) no continuum M contains an uncountable collection of mutually exclusive connected subsets each containing a compact proper set which cuts M; (2) the author's theorem (previously established in the plane) that all save a countable number of the cut points of any continuum M are points of order two (Menger-Urysohn) of M holds true; and indeed that if G is any collection of mutually exclusive connected cuttings of M, then all save a countable number of elements of G can be ϵ-separated by two sets of G; (3) all save a countable number of the *cut points im kleinen* of any continuous curve M are points of order two of M; (4) the junction points of every continuous curve are countable; and (5) if every subcontinuum of a continuous curve M contains uncountably many cut points im kleinen of M, then M is a Menger regular curve. Also the notion of a *junction continuum*, an extension of the notion of a *junction point* due to R. L. Moore, is discussed, and it is pointed out that no *plane continuum* contains an uncountable collection of mutually exclusive junction continua of itself.

17. Professor E. H. Mc Alister: *On the dynamical form of Hooke's law.* Preliminary report.

Hooke's law leads to results not in agreement with experiment when the velocity of strain is not negligible. In particular, the rapid decay of elastic vibrations finds no explanation in Hooke's law, and is accounted for only in small part by the resistance of the air. Kelvin conjectured some internal resistance, and coined the phrase "viscosity of metals," but concluded that the resistance is not merely proportional to the velocity, as in viscous

fluids. This paper attempts to formulate a law suitable for dynamical conditions, as Hooke's law is for statical conditions. The point of mathematical interest is the search for an unknown function by mathematical means, guided by experiment. A form is found which agrees with certain experiments on torsional vibrations of a steel wire. The mathematical investigation indicates that other forms are possible, and the results of this paper are merely tentative, awaiting further research. The paper is therefore presented as a preliminary report. The tentative law admits of isochronous vibrations, in accordance with Rayleigh's dictum, and reduces to Hooke's law when the velocity is null or negligible.

18. Professor D. R. Davis: *The inverse problem of the calculus of variations in space of* $(n+1)$ *dimensions.*

In this paper is considered a system of n differential equations of the form $H_i(x, y_i(x), y_i'(x), y_i''(x))=0$, $(i, j=1, \cdots, n)$, whose solutions are $y_i=y_i(x)$, with the derivatives y_i', y_i''. The conditions required for the given system to have equations of variation which are self-adjoint are found. It is then proved that these conditions are sufficient to insure that a function $f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ can be determined such that the given equations are the differential equations of the solutions for the problem of minimizing the integral

$$I=\int_{x_1}^{x_2} f(x, y_1, \cdots y_n, y_1', \cdots, y_n')dx$$

Incidentally, in this paper, are also found necessary and sufficient conditions that there exist a solution for a system of linear partial differential equations of the first order of the form $B_{iy_i}-B_{iy_j}=\phi_{ij}$, $B_{ix}-A_{y_i}=\phi_i$, $(i, j=1, \cdots, n)$, where $A, B_1, \cdots, B_n, \phi_{ij}, \phi_i$ are functions of $x, y_1, \cdots, y_n$, and $\phi_{ij}=-\phi_{ji}$.

19. Professor G. T. Whyburn: *Concerning certain points of continuous curves.*

In this paper a study is made of the *im kleinen cut points, im kleinen cycle points,* and ramification points (i.e., points of Menger order >2) of a bounded continuous curve M in n dimensions. The point P of M is called an *im kleinen cycle point* of M if for each $\epsilon>0$, M contains a simple closed curve of diameter $<\epsilon$ containing P. Some of the results proved are as follows. Let K, N, H, and W denote the set of all *im kleinen cut points, im kleinen cycle points,* end points, and ramification points respectively of M. Then (1) $K+H+N=M$, (2) $K\cdot N$ and $K\cdot W$ are countable, (3) $K. N\subset W$, (4) if W is dense on an arc t of M, both N and $M-K$ are uncountably dense on t, (5) if M is cyclicly connected and $M-K$ is not uncountably dense on any subcontinuum of M, then (a) W is totally disconnected, and (b) every component of $M-W$ is an arc segment, (6) if $M=K$, then for each $\epsilon>0$, M is the sum of a finite number of ϵ-continua no two having more than one common point.

20. Professor G. T. Whyburn: *Concerning a certain type of regular curve.*

A continuous curve M will be called a *node curve* provided that for each $\epsilon>0$, M is the sum of a finite number of ϵ-continua each having at most two points in common with the rest of M. In this paper a study is made of node curves, and some of the results found are as follows. Let M denote any bounded continuous curve in n dimensions. Then (1) M is a node curve if and only if it is true that if C is any maximal cyclic curve of M then (a) C is a node curve and (b) every point of C which is a limit point of some component of $M-C$ is an *im kleinen cut point* of C; (2) if M is cyclicly connected, M is a node curve without points of order w if and only if no ramification point of M is an *im kleinen cycle point* of M; (3) if no point of M is an *im kleinen cycle point*, then M is a node curve; (4) the ramification points of a node curve are countable and each of them is an *im kleinen cut point.*

21. Professor G. T. Whyburn: *A continuous curve in 3-space.*

In this paper a simple example is given in 3 dimensions of a continuous curve ψ every subcontinuum of which is a continuous curve which has the following properties: (1) ψ contains two points A and B and infinitely many arcs AX_iB ($i=1, 2, 3, \cdots$) from A to B no two of which have any common points except A and B, (2) ψ contains an arcwise connected subset which is not arcwise connected *im kleinen* (in fact, not connected *im kleinen* at all) and which has an inaccessible boundary point, and (3) ψ contains a connected subset which is not arcwise connected. It is already known that ψ could not lie in the plane and have properties (1) or (2). It is also pointed out in this paper that no Menger regular curve can have properties (1) or (2), and a number of related questions are settled in n dimensions for regular curves and for continuous curves all of whose subcontinua are continuous curves.

22. Mrs. E. N. Williamson: *Two plane curves in point correspondence.*

This paper deals with the projective differential properties of two plane curves in one-to-one reciprocal point correspondence, making use of a defining system of three ordinary linear homogeneous differential equations in two dependent variables, two of the equations being of the second order and one of the first. Geometric interpretations are obtained for certain of the invariants and covariants of the system.

23. Dr. L. Ayres, (National Research Fellow): *Concerning the arc-curves and the basic sets of a continuous curve.* Second Paper.

This paper studies the relations between the cyclic elements of a continuous curve and the arc-curves, and develops some new properties of the

basic sets of a continuous curve. It is shown that every arc-curve of more than one point is a collection of cyclic elements, and every connected collection of cyclic elements is an arc-curve. Every arc-curve of two points is a simple cyclic chain and conversely. A subset K of a continuous curve M, which is not cyclicly connected, is an irreducible basic set of M if and only if (1) no point of K is a cut point of M, (2) every point of K belongs to some node of M and each node of M contains exactly one point of K. An irreducible basic set of a continuous curve M contains the set of all end points of M, and an irreducible basic set has many of the properties of the set of all end points.

B. A. BERNSTEIN.
Secretary of the Section

ON A CERTAIN SYSTEM OF ∞^{r-2} LINES IN r-SPACE

BY B. C. WONG

This paper deals with the following theorem.

The locus of ∞^{r-2} lines incident with r given $(r-2)$-spaces in S_r is an $(r-1)$-dimensional manifold V_{r-1}^{r-1} of order $r-1$.

To prove this, we note that for $r=2$ there is one line joining two given points in a plane and that for $r=3$ the lines meeting three given lines in an S_3 form a quadric surface. If $r=4$, that is, if four planes are given in S_4, the locus of the ∞^2 lines incident with them is a V_3^3. This cubic hypersurface with its many interesting properties has been studied by a number of writers.*

If $r=5$, that is, if five three-spaces are given in S_5, pass an S_4 through one of them, say R_3. This S_4 meets the other four 3-spaces in four planes and the lines incident with these four planes are also incident with R_3. These lines form a V_3^3 in S_4. The manifold of the ∞^3 lines incident with the five given 3-spaces is intersected by the S_4 through R_3 in R_3 and a V_3^3 and is therefore of order 4. If we apply this process of reasoning to the cases $r=6$, 7, etc., we soon arrive at the general theorem stated above.

Consider another proof. Let the r given $(r-2)$-spaces in S_r be S'_{r-2}, S''_{r-2}, $\cdots$, $S^{(r)}_{r-2}$, and further let a general line l be given. The points of l determine with S'_{r-2}, S''_{r-2}, $\cdots$, $S^{(r-1)}_{r-2}$ $r-1$ projective pencils of hyperplanes. The rth $(r-2)$-space, $S^{(r)}_{r-2}$, intersects these pencils in $r-1$ pencils of $(r-3)$-spaces. As there are $r-1$ sets of corresponding $(r-3)$-spaces each intersecting in a point, there are $r-1$ lines of intersection of corresponding hyperplanes of the pencils in S_r which meet $S^{(r)}_{r-2}$. Hence, the general line l meets $r-1$ of the ∞^{r-2} lines

* See Bertini, *Projecktive Geometrie Mehrdimensionaler Räume*, 1924, Chap. 8, §§25–36, where references are given.

incident with the r given $S_{r-2}^{(i)}[i=1, 2, \cdots, r]$, and therefore the manifold of the ∞^{r-2} lines is of order $r-1$.

It is of interest to note that the V_{r-1}^{r-1} in question contains the r given $(r-2)$-spaces $S_{r-2}^{(i)}$ $[i=1, 2, \cdots, r]$. These $S_{r-2}^{(i)}$ intersect ν by ν in $\binom{r}{\nu}$ $(r-2\nu)$-spaces $S_{r-2\nu}^{(j)}$ $[\nu=1, 2, \cdots, r/2$ if r is even, $\nu=1, 2, \cdots, (r-1)/2$ if r is odd; $j=1, 2, \cdots, \binom{r}{\nu}]$. The V_{r-1}^{r-2} contains all these $S_{r-2\nu}^{(j)}$ ν-ply. Any V_{r-1}^{r-1} in S_r containing r $(r-2)$-spaces is a hypersurface of this type, for any line meeting these r $(r-2)$-spaces lies entirely on the hypersurface and there are ∞^{r-2} such. For the case $r=4$, there are additional planes and conical points. The V_3^3, besides containing the four given planes and the six conical points in which they intersect two by two, contains eleven other planes and four other conical points. These fifteen planes and ten nodes are such that each plane contains four nodes and each node is on six planes.

The equation of the V_{r-1}^{r-1} we are considering is not yet known. The writer has derived the following equation for V_3^3 in S_4:

$$x_0x_1x_4 + x_1x_2x_3 + x_2x_3x_4 - x_0x_2x_3 - x_1x_2x_4 - x_1x_3x_4 = 0.$$

The ten conical points are the vertices of the coordinate simplex, the unit point and the following:

(1:1:1:0:0), (1:0:1:0:1), (1:0:0:1:1), (1:1:0:1:0).

By the linear transformation

$$\begin{aligned}
\rho x_0 &= & -y_2 & & +y_3, \\
\rho x_1 &= y_0 + y_1 & & +y_3 & +y_4, \\
\rho x_2 &= & y_1 & +y_3, & \\
\rho x_3 &= y_0 & & +y_3, & \\
\rho x_4 &= & -y_2 & & -y_4
\end{aligned}$$

the above equation is transformed into

$$y_0^3 + y_1^3 + y_2^3 + y_3^3 + y_4^3 + y_5^3 = 0,$$

where

$$y_0 + y_1 + y_2 + y_3 + y_4 + y_5 = 0.$$

THE UNIVERSITY OF CALIFORNIA

NEW DIVISION ALGEBRAS

BY L. E. DICKSON

1. *Introduction.* No technical acquaintance with linear algebras is presupposed in this note. We consider only linear algebras for which multiplication is associative. As with quaternions, an algebra A is called a *division* algebra if every element $\neq 0$ of A has an inverse in A. A division algebra A over a field F is called *normal* if the numbers of F are the only elements of A which are commutative with every element of A.

In a paper recently offered to the Transactions of this Society, A. A. Albert determined all normal division algebras of order 16 and found a new type. The object of this note is to derive from mild assumptions the corresponding type of normal division algebras A of order $4p^2$, where p is a prime. We shall first draw simple conclusions from an initial assumption.*

Assumption 1. Let A contain an element i_1 satisfying an equation $f(\omega^2)=0$ of degree $2p$ with only even powers of ω, whose coefficients are in F, that of ω^{2p} being unity, and which is irreducible in F, such that the polynomials in i_1 are the only elements of A which are commutative with every element of A.

2. LEMMA 1. *Let an element i_2 of A be commutative with $I=i_1^2$, but not with i_1 itself. The algebra S generated by i_1 and i_2 is of order $4p$. It may be regarded as an algebra of order 4 with the basis $1, i_1, i_2, i_1i_2$ over $F(I)$; this algebra is normal. In other words, the polynomials in I are the only elements of S which are commutative with every element of S.*

Let K denote the field composed of all those elements of

* Except for the requirement concerning even powers of ω, Assumption 1 is proved in the writer's *Algebren und ihre Zahlentheorie*, Zürich, 1927, pp. 262–3.

S which are commutative with every element of S. If K is of order k and S is of order s over F, then S is a normal division algebra of order n^2 over K, where $s=n^2k$. Since K contains the root I of an equation of degree p irreducible in F, the subfield $F(I)$ is of order p, whence k is a multiple of p.

Since i_2 is not commutative with i_1, i_2 is not a polynomial in i_1 and hence is not a rational function of i_1. Thus

$$(1) \qquad i_1^j,\ i_1^j i_2, \qquad (j=0,1,\cdots,2p-1),$$

are linearly independent with respect to F. Hence $s\geqq 4p$. Since S and A are normal over different fields K and F, $S\neq A$. Thus s is a divisor $<4p^2$ of $4p^2$. First, let $p>2$. If s is not divisible by p^2, then $s=4p$. But if s is divisible by p^2, either $s=2p^2$, or $s=p^2$ and $p>4$. If $p=2$, evidently $s=8=4p$.

If either $s=p^2$, $p>4$, or $s=2p^2$, $p>2$, then $s=n^2k$ and the divisibility of k by p show that $n=1$, $S=K$, contrary to the fact that i_2 is not commutative with i_1.

Hence $s=4p=n^2k$, whence $n=2$, $k=p$. Thus $K=F(I)$ and S is a normal algebra of order 4 over $F(I)$. The $4p$ elements (1) form a basis of S over F.

3. LEMMA 2. *Any element of A which is commutative with $I=i_1^2$ belongs to S.*

Any element not in S extends S to a division subalgebra whose order exceeds $4p$, is a multiple of $4p$, and is a divisor of $4p^2$. Hence it extends S to A itself (of order $4p^2$).

Suppose that e is commutative with I and is not in S. Since I is commutative with every element of S and with e, which extends S to A, I is commutative with every element of A. Since I is not in F, this contradicts the hypothesis that A is normal over F.

4. *Assumption* 2. Let A contain elements i_1 and z such that i_1 satisfies Assumption 1 and such that

$$(2) \qquad i_2=zi_1z^{-1},\ i_3=zi_2z^{-1},\ \cdots,\ i_p=zi_{p-1}z^{-1}$$

are all commutative with $I=i_1^2$, while i_2 is not commutative with i_1, and $i_2^2 \neq I$.

Since $zIz^{-1}=i_2^2 \neq I$, z is not commutative with I and hence is not in S. By §3, z extends S to A. Since (1) gives a basis of S, every element of S is of the form

$$G = p(i_1) + q(i_1)i_2. \tag{3}$$

Then

$$G' = zGz^{-1} = p(i_2) + q(i_2)i_2. \tag{4}$$

For $p \geqq 3$, i_2 is commutative with i_1^2 and hence is in S. Thus

$$zG = G'z, \ G' \text{ in } S. \tag{5}$$

5. LEMMA 3. *$i_1^2, \cdots, i_p^2$ are all distinct.*

Suppose that $i_{r+1}^2 = i_1^2$, where r is one of $2, 3, \cdots, p-1$. Then

$$z^r i_1^2 z^{-r} = i_{r+1}^2 = i_1^2,$$

whence z^r is commutative with i_1^2 and is in S. Using also (5), we see that every element of the algebra A obtained by extending S by z is of the form

$$H_0 + H_1 z + \cdots + H_{r-1}z^{r-1},$$

where each H is in S. Since S is of order $4p$, the order of A is $\leqq 4p \cdot r < 4p^2$. But A is of order $4p^2$.

Suppose that $i_{r+s}^2 = i_s^2$ $(r>0, s>1)$. These are the transforms of i_{r+s-1}^2 and i_{s-1}^2 by z. Hence the latter are equal. After $s-1$ such steps, we get $i_{r+1}^2 = i_1^2$, just proved impossible.

6. LEMMA 4. *We have the following identity*:

$$f(\epsilon) \equiv (\epsilon - i_p^2) \cdots (\epsilon - i_2^2)(\epsilon - i_1^2). \tag{6}$$

Note that

$$i_r \text{ is commutative with } i_{r+1}, \cdots, i_p, \quad (r=1, \cdots, p-1). \tag{7}$$

This is true by Assumption 2 if $r=1$. To proceed by induction, let (7) hold when $r=j$, whence i_j^2 is commutative

with i_k for $k \geqq j+1$. Transformation by z shows that $i_j^2{}_{+1}$ is commutative with i_{k+1}, whence (7) holds when $r=j+1$.

Write v_j for i_j. As a special case of (7), $v_1, \cdots, v_p$ are commutative. The indeterminate ϵ is commutative with every quantity of A. Thus z transforms $f(\epsilon)$ into itself. But $f(v_1)=0$. Hence by (2), $f(v_2)=0, \cdots, f(v_p)=0$. Let

$$f(\epsilon) = \sum_{j=0}^{p} a_j \epsilon^{p-j}, \quad q(\epsilon) = \sum_{j=0}^{p-1} c_j \epsilon^{p-1-j}, \quad a_0 = c_0 = 1,$$

$$c_j = a_j + c_{j-1} v_1, \qquad (j = 1, \cdots, p).$$

Then, since v_1 is commutative with ϵ,

$$f(\epsilon) \equiv q(\epsilon)(\epsilon - v_1) + c_p. \tag{8}$$

By induction on r,

$$c_r = \sum_{j=0}^{r} a_j v_1^{r-j}, \qquad c_p = f(v_1) = 0.$$

Since v_i is commutative with v_1, we obtain a true equality from (8) by replacing ϵ by v_i. Thus $0 = q(v_i)(v_i - v_1)$. The second factor is not zero if $i \geqq 2$. In our division algebra we therefore have $q(v_i) = 0$ when $i \geqq 2$.

We may repeat this argument with f and v_1 replaced by q and v_2. Hence $q(\epsilon) \equiv r(\epsilon)(\epsilon - v_2)$, in which the coefficients of $r(\epsilon)$ are polynomials in v_1 and v_2. Since they are commutative with v_j, $0 = r(v_j)(v_j - v_2)$. Hence $r(v_j) = 0$ when $j \geqq 3$.

Proceeding similarly, we ultimately obtain

$$f(\epsilon) \equiv (\epsilon - v_p) \cdots (\epsilon - v_2)(\epsilon - v_1).$$

7. THEOREM 1. *$f(\epsilon) = 0$ is a cyclic equation.*

By (6), $i_1^2 + \cdots + i_p^2$ is a number of F and hence is transformed into itself by z. But z transforms i_1^2 into i_2^2, $\cdots$, i_{p-1}^2 into i_p^2. Hence z must transform i_p^2 into i_1^2. Since z^{p-2} transforms i_2^2 into i_p^2, z^{p-1} transforms i_2^2 into i_1^2 and evidently transforms i_1 into i_p. Hence z^{p-1} transforms

$i_2^2 i_1$ and $i_1 i_2^2$ into $i_1^2 i_p$ and $i_p i_1^2$. The latter are equal by by Assumption 2. Hence the former are equal. Since i_2^2 is therefore commutative with both generators i_1 and i_2 of S, it is commutative with every element of S. By Lemma 1, $i_2^2 = \theta(i_1^2)$, where θ is a polynomial with coefficients in F. Transformation by z gives

$$i_3^2 = \theta(i_2^2) = \theta[\theta(i_1^2)] = \theta^2(i_1^2),$$

if $\theta^r(k)$ denotes the rth iterative of $\theta(k)$ and not its rth power. By induction,

$$i_{r+1}^2 = \theta^r(i_1^2). \tag{9}$$

Take $r = p-1$ and transform by z. Hence

$$i_1^2 = \theta^{p-1}(i_2^2) = \theta^p(i_1^2). \tag{10}$$

Since $f(\epsilon) = 0$ has these properties, it is cyclic.

8. THEOREM 2. *Every element of A can be expressed in one and only one way in the form*

$$A_0 + A_1 z + \cdots + A_{p-1} z^{p-1}, \tag{11}$$

where each A_j is in S. The product any two sums (11) *can be expressed as a third such sum by means of*

$$zG = G'z, \quad z^p = s, \tag{12}$$

where G, G', s are all in S and are defined in (4), (5).

Since z^{p-1} transforms i_1^2 into i_p^2, and z transforms the latter into the former, z^p is commutative with i_1^2 and hence is in S. By means of (12), every element of A (to which z extends S) can be expressed in the form (11). Since S and A are of orders $4p$ and $4p^2$, two polynomials (11) are distinct unless identical.

9. THEOREM 3. *S is an algebra of generalized quaternions over $F(I)$ with the basis* $1, i_1, y, i_1 y$, *where* $y = i_1 i_2 - i_2 i_1$.

Since i_2 is not commutative with i_1, $y \neq 0$. Since i_2 is commutative with i_1^2,

$$yi_1 = -i_1 y. \tag{13}$$

Thus y is not commutative with i_1 and hence is not a polynomial in i_1. We may therefore replace the basis (1) of S over F by i_1^j, $i_1^j y$. Thus S has the basis in Theorem 3.

By §7, i_2^2 is commutative with i_1. Hence

$$r = i_1 i_2 + i_2 i_1$$

is commutative with i_2. Since i_2 is commutative with $I = i_1^2$, $ri_1 = i_1 r$. Hence r is commutative with every element of S. Thus r is a polynomial $P(I)$ in I. We have

$$2i_1 i_2 = P(I) + y, \quad 2i_2 i_1 = P(I) - y.$$

But y is commutative with I. Hence

$$4i_1 i_2^2 i_1 = P^2 - y^2.$$

Since i_2^2 is commutative with i_1,

$$y^2 = [P(I)]^2 - 4I\,\theta(I).$$

This fact that y^2 is a polynomial in I and relation (13) together show that S is an algebra of generalized quaternions over $F(I)$.

THE UNIVERSITY OF CHICAGO

NOTE ON THE DERIVATIVE CIRCULAR CONGRUENCE OF A POLYGENIC FUNCTION*

BY EDWARD KASNER

In the first paper† I published on polygenic functions I stated the following theorem.

If

$$w = \phi(x, y) + i\psi(x, y)$$

is a polygenic function of

$$z = x + iy$$

then the first derivative of w with regard to z, $dw/dz = \gamma = \alpha + i\beta$, is represented in the γ-plane by the congruence of circles

$$(\alpha - H)^2 + (\beta - K)^2 = R^2 \tag{1}$$

where

$$\begin{cases} 2H = \phi_x + \psi_y, & 2K = -\phi_y + \psi_x, \\ R^2 = h^2 + k^2, \\ 2h = \phi_x - \psi_y, & 2k = \phi_y + \psi_x. \end{cases} \tag{2}$$

To every point (x, y) of the z-plane corresponds by means of dw/dz that circle of (1) determined by the particular pair of values (x, y).

In this paper I wish to study the converse problem. Let $H(x, y)$, $K(x, y)$, and $R(x, y)$ be three arbitrary functions of (x, y)‡ and for every pair of values (x, y) construct in the γ-plane the circle whose center is (H, K)

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† *A new theory of polygenic functions*, Science, vol. 66 (Dec., 1927), pp. 581–582. See also a previous paper by Hedrick, Ingold, and Westfall, Journal de Mathématiques, *Theory of non-analytic functions of a complex variable*, (6), vol. 2 (1923), pp. 327–342.

‡ We require, however, that these functions be continuous and have continuous first, second, and third derivatives.

and whose radius is R. What are the necessary and sufficient conditions to be fulfilled by H, K, and R, in order that the congruence of circles so obtained may map the derivative of a polygenic function? In other words, when can we find two functions $\phi(x, y)$ and $\psi(x, y)$, such that the relations (2) expressing the correspondence between the circles of the congruence and the points of the z-plane are fulfilled?

The required conditions are found by eliminating ϕ and ψ from the three equations

$$(3)\quad \begin{cases} 2H = \phi_x + \psi_y, \qquad 2K = -\phi_y + \psi_x, \\ 4R^2 = (\phi_x - \psi_y)^2 + (\phi_y + \psi_x)^2. \end{cases}$$

This is accomplished by forming all the first derivatives, then all the second derivatives, then all the third derivatives etc., of system (3), till at one step the total number of equations obtained is larger than the number of derivatives $\phi_x, \phi_y, \psi_x, \cdots$ occurring in them. The necessary and sufficient conditions on H, K, R are then furnished by the eliminants with regard to $\phi_x, \phi_y, \psi_x, \cdots$ of the entire system of equations.

Calculation shows that the process of differentiation has to be carried up to the third derivatives of (3); the total number of equations obtained is then thirty, while the number of derivatives $\phi_x, \phi_y, \psi_x, \cdots, \cdots \psi_{yyyy}$ is only twenty-eight. The system will therefore have two eliminants, that is, H, K, *and* R *have to obey two conditions in order that their congruence may represent the derivative of a polygenic function.*

We will not carry out the elimination directly from (3) but from another system of three independent functions of H, K and R, namely,

$$M = \tfrac{1}{2}[H_x - K_y + i(H_y + K_x)] = \tfrac{1}{2}[h_x + ik_x - i(h_y + ik_y)],$$
$$N = \tfrac{1}{2}[H_x - K_y - i(H_y + K_x)] = \tfrac{1}{2}[h_x - ik_x + i(h_y - ik_y)],$$
$$O = R^2 = h^2 + k^2.$$

We simplify this system by the following changes.

1. We use

$$u = x + iy, \quad v = x - iy^*$$

as independent variables instead of x and y; in order to be able to substitute

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2i}$$

into ϕ and ψ, we assume these functions to be analytic, that is, developable into power series, in x and y in the region considered.†

2. We use

$$h + ik = a, \quad h - ik = b$$

instead of ϕ and ψ as the functions to be eliminated. Then O, M, N assume the simple forms

$$O = a \cdot b, \quad M = a_u, \quad N = b_v.$$

To carry through the elimination of a and b we start out from the following expressions:

$$(4) \quad \begin{cases} M = a_u, & O = ab, \\ N = b_v, & O_{uv} = a_{uv}b + ab_{uv} + a_u b_v + a_v b_u, \\ O_u = a_u b + ab_u, & M_v = a_{uv}, \\ O_v = a_v b + ab_v, & N_u = b_{uv}. \end{cases}$$

From these eight equations we eliminate the six first and second derivatives of a and b which occur in them. The resulting equations are

$$(5) \quad ab - O = 0, \quad Aa + Bb + C = 0,$$

where

$$(6) \quad \begin{cases} A = ON_u - O_u N, \\ B = OM_v - O_v M, \\ C = O_u O_v + O(2MN - O_{uv}). \end{cases}$$

* We use u and v instead of z and $\bar{z}$, merely to simplify the writing when these letters occur as subscripts.

† In my previous papers, ϕ and ψ were only assumed to be continuous differentiable functions of x and y.

Differentiating the second equation of (5) with regard to u and with regard to v, and replacing the first derivatives of a and b in the two expressions obtained by their values in M, N, O_u, O_v, a, b according to (4), we find

$$(7) \quad \begin{cases} A_u a^2 + (AM + C_u)a - BMb + (BO)_u = 0, \\ B_v b^2 + (BN + C_v)b - ANa + (AO)_v = 0. \end{cases}$$

The elimination of a and b from (5) and (7) furnishes two equations in O and its derivatives up to the third order, and in M and N and their derivatives up to the second order. If in these equations we return from the complex quantities u, v, M, N to the original real quantities x, y, H, K, and from O to R, we obtain two equations containing R and the derivatives of H, K, and R up to the third order. According to what was said above, they represent the necessary and sufficient conditions that must be fulfilled by H, K, and R.

These two equations will be either mutually conjugate or each self-conjugate. This follows from the fact that u and v, a and b, M and N, A and B, are mutually conjugate, while O and C are self-conjugate, so that equations (5) are self-conjugate, while equations (7) are mutually conjugate.

The elimination of a and b from (5) and (7) can be carried through as follows. By multiplying the linear equation of (5) first by $A_u \cdot a/A$, then by $B_v \cdot b/B$, and subtracting it first from the first, then from the second, equation of (7), we reduce (5) and (7) to the simpler system

$$(8) \quad \begin{cases} ab - O = 0, & Aa + Bb + C = 0, \\ A_1 a + B_1 b + C_1 = 0, & A_2 a + B_2 b + C_2 = 0, \end{cases}$$

where

$$A_1 = A(AM + C_u) - A_u C, \quad A_2 = -ABN,$$

$$B_1 = -ABM, \quad B_2 = B(BN + C_v) - B_v C,$$

$$C_1 = A(BO)_u - BA_u O, \quad C_2 = B(AO)_v - AB_v O.$$

Then

$$(9)\qquad \begin{vmatrix} A & B & C \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix} = 0, \quad \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} \cdot \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}^2 \cdot O$$

represent the two eliminants of the system (8).

The two necessary and sufficient conditions to be fulfilled by three arbitrary functions $H(x, y)$, $K(x, y)$, and $R(x, y)$ of x and y in order that the congruence of circles

$$(\alpha - H)^2 + (\beta - K)^2 = R^2$$

which they determine in the (α, β)-plane may represent the derivative of a polygenic function are obtained by retransforming the two equations (9) from the quantities u, v, M, N, O into the quantities x, y, H, K, R; the two final conditions contain R and the derivatives of H, K and R up to the third order.

COLUMBIA UNIVERSITY

ON THE INVERSION OF ANALYTIC TRANSFORMATIONS*

BY B. O. KOOPMAN†

We wish to consider the transformation

$$(1)\qquad x_i = f_i(y_1, \cdots, y_n), \qquad (i = 1, \cdots, n),$$

in the neighborhood of the origin $(y) = (0)$, at which point the functions f_i are analytic, and vanish simultaneously. We are interested in the case in which the jacobian

$$J = \frac{\partial(f_1, \cdots, f_n)}{\partial(y_1, \cdots, y_n)}$$

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* Presented to the Society under the title *Some theorems on inversion*, October 29, 1927.

The methods and point of view of the second chapter of W. F. Osgood's *Lehrbuch der Funktionentheorie* are assumed throughout this paper. Our results may be regarded as the extension of §20 of that work.

† National Research Fellow, 1926-1927.

is zero at the origin, without vanishing identically. We shall assume throughout that $(y)=(0)$ is a non-specialized point of the locus $J=0$; that is, it will be sufficient for our results to apply only when $(y)=(0)$ does not belong to certain exceptional $(2n-4)$-dimensional loci satisfying a further equation $E(y_1, \cdots, y_n)=0$, where E is analytic and relatively prime to J.

What we wish to accomplish is to obtain, with the aid of non-singular analytic transformations of the variables (x) and the variables (y), a standard form for the transformation (1), having particularly in view the investigation of its inverse.

Since $(y)=(0)$ is a non-specialized point of $J=0$, we shall have in a neighborhood of this point

$$J(y_1, \cdots, y_n) \equiv S(y_1, \cdots, y_n)^{\rho} \cdot L(y_1, \cdots, y_n),$$

where S and L are analytic at the origin, L does not vanish there, while S has there the value zero. Moreover, at least one derivative, say $\partial S/\partial y_1$, will be distinct from zero for $(y)=(0)$.

After the non-singular transformation

$$y_1' = S(y_1, \cdots, y_n), \qquad (y_2' = y_2, \cdots, y_n' = y_n),$$

the equations (1) retain the form

$$(1') \qquad x_i = f_i'(y_1', \cdots, y_n'), \qquad (i = 1, \cdots, n),$$

while their jacobian is readily found to be

$$J'(y_1', \cdots, y_n') \equiv y_1'^{\rho} \cdot L'(y_1', \cdots, y_n'),$$

L' being of the same nature as L. Hence we may and we shall assume that (1) has this form, that is, we shall drop the accents.

We now introduce, with Osgood, the matrix

$$(2) \qquad \left\| \begin{matrix} \dfrac{\partial f_1}{\partial y_2}, & \cdots, & \dfrac{\partial f_n}{\partial y_2} \\ \cdot & \cdots & \cdot \\ \dfrac{\partial f_1}{\partial y_n}, & \cdots, & \dfrac{\partial f_n}{\partial y_n} \end{matrix} \right\|.$$

Suppose, in the first place, that when $y_1=0$, every element vanished identically. Considering the Taylor expansions of the f's, and observing that they are zero for $(y)=(0)$, we see that

$$f_i = y_1^{p_i}F_i(y_1, \cdots, y_n), \qquad (p_i > 0\,;\ i = 1, \cdots, n).$$

Here the F's are analytic at the origin, and are not identically zero when $y_1=0$. Since the origin is a non-specialized point, we assume that $F_1 \neq 0$ there. Then the transformation

$$\begin{cases} y_1' = y_1[F_1(y_1, \cdots, y_n)]^{1/p_1}, \\ y_i' = y_i, \qquad\qquad (i = 1, \cdots, n), \end{cases}$$

is analytic and non-singular at $(y)=(0)$; and we have

$$\begin{cases} f_1(y) = f_1'(y') = y_1'^{p_1}, \\ f_i(y) = f_i'(y') = y_1'^{p_i}F_i(y_1', \cdots, y_n'), \quad (i = 2, \cdots, n). \end{cases}$$

Hence, dropping accents, we may assume (1) to have the form

$$(3) \qquad \begin{cases} x_1 = y_1^{p_1}, \\ x_i = y_1^{p_i}F_i(y_1, \cdots, y_n), \qquad (i = 2, \cdots, n). \end{cases}$$

In the case where the elements of (1) do not behave in the above manner, there will be a determinant j of (2) of order $r(0<r<n)$, which does not vanish identically when $y_1=0$, while every determinant in (2) of greater order is zero identically for $y_1=0$. Let us assume, as we may, that

$$j = \frac{\partial(f_{n-r+1}, \cdots, f_n)}{\partial(y_{n-r+1}, \cdots, y_n)}.$$

The origin being non-specialized, we assume that $j \neq 0$ when $(y)=(0)$.

The last r of the equations (1) admit the unique solutions

$$\begin{cases} y_{n-r+1} = g_{n-r+1}(y_1, \cdots, y_{n-r}\,;\, x_{n-r+1}, \cdots, x_n), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n = g_n(y_1, \cdots, y_{n-r}\,;\, x_{n-r+1}, \cdots, x_n), \end{cases}$$

where the g's are analytic for zero values of the arguments, and vanish for these values. When these expressions are substituted into the first $n-r$ equations (1), we obtain

$$
\begin{aligned}
x_i = {} & f_i[y_1, \cdots, y_{n-r}; g_{n-r+1}(y_1, \cdots, y_{n-r}; x_{n-r+1}, \\
(4) \qquad & \cdots, x_n), \cdots, g_n(y_1, \cdots, y_{n-r}; x_{n-r+1}, \\
& \cdots, x_n)], \\
= {} & H_i(y_1, \cdots, y_{n-r}; x_{n-r+1}, \cdots, x_n), \\
& (i = 1, \cdots, n-r).
\end{aligned}
$$

I say that, when $y_1 = 0$, the H's do not contain the y's:

$$
\frac{\partial H_i}{\partial y_j} \equiv 0 \text{ for } y_1 = 0; \qquad (i = 1, \cdots, n-r;\ j = 1, \cdots, n-r).
$$

This follows, as in the proof of the theorem of functional dependence, from the fact that, when $y_1 = 0$, all determinants of order $r+1$ of (2) are identically zero.* It follows that the equations (4) have the form

$$
\begin{aligned}
x_i = {} & G_i(x_{n-r+1}, \cdots, x_n) \\
& + y_1^{p_i}\Gamma_i(y_1, \cdots, y_{n-r}; x_{n-r+1}, \cdots, x_n),
\end{aligned}
$$

for $p_i > 0$; $i = 1, \cdots, n-r$, where G and Γ are analytic, and $G_i(0) = 0$.

Now perform the non-singular analytic transformation

$$
\begin{cases}
x_i' = x_i - G_i(x_{n-r+1}, \cdots, x_n), & (i = 1, \cdots, n-r), \\
x_j' = x_j, & (j = n-r+1, \cdots, n).
\end{cases}
$$

The equations (1) retain their form, becoming

$$
(1'') \qquad x_i' = f_i'(y_1, \cdots, y_n), \qquad (i = 1, \cdots, n);
$$

and

$$
J' = \frac{\partial(x_1', \cdots, x_n')}{\partial(y_1, \cdots, y_n)} = \frac{\partial(x_1', \cdots, x_n')}{\partial(x_1, \cdots, x_n)} \frac{\partial(x_1, \cdots, x_n)}{\partial(y_1, \cdots, y_n)},
$$

* See Osgood, *Funktionentheorie*, vol. 2, Chap. 2, §23.

and hence $J' = J = y_1^e L$; also, when $y_1 = 0$, $j' = j$. Indeed, the properties which we have demonstrated for equation (1) are retained. But it is seen at once that $f_1', \cdots, f_{n-r}'$ all vanish identically when $y_1 = 0$. Hence, dropping the accents, we may write them in the form

$$f_i = y_1^{p_i} F_i(y_1, \cdots, y_n), \quad (p_i > 0\,;\ i = 1, \cdots, n - r).$$

Reasoning as before, we may make the non-singular transformation

$$\begin{cases} y_1' = y_1 [F_1(y_1, \cdots, y_n)]^{1/p_1}, \\ y_i' = y_i, \qquad\qquad (i = 2, \cdots, n), \end{cases}$$

and, dropping the accents, we may write the equations in the form

$$\begin{cases} x_1 = y_1^{p_1}, \\ x_2 = y_1^{p_2} F_2(y_1, \cdots, y_n), \\ \cdots\cdots\cdots\cdots \\ x_{n-r} = y_1^{p_{n-r}} F_{n-r}(y_1, \cdots, y_n), \\ x_i = f_i(y_1, \cdots, y_n), \\ \qquad\qquad (i = n - r + 1, \cdots, n). \end{cases}$$

A little computation shows that the new determinant j does not vanish for $(y) = (0)$. Hence we may apply the following non-singular transformation

$$\begin{cases} y_i' = y_i, \qquad\qquad (i = 1, \cdots, n - r), \\ y_j' = f_j(y_1, \cdots, y_n), \quad (j = n - r + 1, \cdots, n), \end{cases}$$

and, dropping the accents, we may write the equations in the form

$$(5) \qquad \begin{cases} x_1 = y_1^{p_1}, \\ x_2 = y_1^{p_2} \phi_2(y_1, \cdots, y_n), \\ \cdots\cdots\cdots\cdots \\ x_{n-r} = y_1^{p_{n-r}} \phi_{n-r}(y_1, \cdots, y_n), \\ x_i = y_i, \qquad (i = n - r + 1, \cdots, n). \end{cases}$$

These equations, which include (3) as a special case ($r=0$), will be made the basis of the rest of our work.

To begin with, we form the square matrix

$$(6) \qquad \left\| \begin{matrix} \dfrac{\partial \phi_2}{\partial y_2} & \cdots & \dfrac{\partial \phi_{n-r}}{\partial y_2} \\ \cdot & \cdots & \cdot \\ \dfrac{\partial \phi_2}{\partial y_{n-r}} & \cdots & \dfrac{\partial \phi_{n-r}}{\partial y_{n-r}} \end{matrix} \right\|.$$

Consider first the case where all the elements are identically zero when $y_1=0$. Then the ϕ's have the form

$$\phi_i = \psi_i(y_{n-r+1}, \cdots, y_n) + y_1\phi_i'(y_1, \cdots, y_n), \qquad (i = 2, \cdots, n-r).$$

In this case we form the matrix (6′), in which ϕ_i' replaces ϕ_i in (6). If all the elements of (6′) are identically zero for $y_1=0$, we repeat the process. Since the right-hand members of (5) actually contain *all* the y's (for $J\not\equiv 0$), we are eventually stopped; that is, we have, for a certain integer λ,

$$\begin{cases} \phi_i = \psi_i + \psi_i' y_1 + \cdots + \psi_i^{(\lambda-1)} y_1^{\lambda-1} \\ \qquad\quad + y_1^{\lambda}\phi_i^{(\lambda)}(y_1, \cdots, y_n), \\ \psi_i^{(\kappa)} = \psi_i^{(\kappa)}(y_{n-r+1}, \cdots, y_n), \quad (i = 2, \cdots, n-r), \end{cases}$$

where the matrix $(6^{(\lambda)})$ of the $\phi^{(\lambda)}$'s does not have all its elements identically zero when $y_1=0$.

In all cases, we are led to a matrix, (6) or $(6^{(\lambda)})$, in which at least one determinant δ of order $\rho(0<\rho<n-r)$ does not vanish identically for $y_1=0$, whereas every determinant of higher order is identically zero for that value. Let us assume, as we may, that

$$\delta = \frac{\partial(\phi_{n-r-\rho+1}, \cdots, \phi_{n-r})}{\partial(y_{n-r-\rho+1}, \cdots, y_{n-r})},$$

and that $\delta\not\equiv 0$ at the non-specialized point $(y)=(0)$. We perform the non-singular substitution

$$\left\{\begin{array}{l} y_1' = y_1, \cdots, y_{n-r-\rho}' = y_{n-r-\rho}, \\ y_{n-r-\rho+1}' = \phi_{n-r-\rho+1}(y) - \phi_{n-r-\rho+1}(0), \\ \quad \cdots \cdots \cdots \cdots \\ y_{n-r}' = \phi_{n-r}(y) - \phi_{n-r}(0), \\ y_{n-r+1}' = y_{n-r+1}, \cdots, y_n' = y_n. \end{array}\right.$$

This reduces (5) to a transformation which, dropping the accents, we may write in the form

$$\left\{\begin{array}{l} x_1 = y_1^{p_1}; \\ x_i = \omega_{i0} y_1^{p_i} + \omega_{i1} y_1^{p_i+1} + \cdots + \omega_{i,\lambda+\mu} y_1^{p_i+\lambda+\mu} + y_1^{p_i+\lambda+\mu}\Phi_i, \\ \qquad\qquad (i = 2, \cdots, n-r-\rho), \\ x_j = \omega_{j0} y_1^{p_j} + \omega_{j1} y_1^{p_j+1} + \cdots + \omega_{j\lambda} y_1^{p_j+\lambda} y_j, \\ \qquad\qquad (j = n-r-\rho+1, \cdots, n-r), \\ x_k = y_k, \qquad\qquad (k = n-r+1, \cdots, n). \end{array}\right.$$

Here $\omega_{i\beta} = \omega_{i\beta}(y_{n-r-\rho+1}, \cdots, y_n)$, $\omega_{j\beta} = \omega_{j\beta}(y_{n-r+1}, \cdots, y_n)$, when $\beta > 0$; otherwise, the ω's are constants. We repeat the discussion of the last two paragraphs, applying it to $\Phi_2, \cdots, \Phi_{n-r-\rho}$, and the matrix

$$\left\| \begin{array}{ccc} \dfrac{\partial \Phi_2}{\partial y_2} & \cdots & \dfrac{\partial \Phi_{n-r-\rho}}{\partial y_2} \\ \cdots\cdots\cdots & & \\ \dfrac{\partial \Phi_2}{\partial y_{n-r-\rho}} & \cdots & \dfrac{\partial \Phi_{n-r-\rho}}{\partial y_{n-r-\rho}} \end{array} \right\|,$$

in place of $\phi_2, \cdots, \phi_{n-r}$ and (6) or $(6^{(\lambda)})$. Continuing in this manner, we find, at the end of a finite number of steps, the final standard form for the equations (1) of the transformation we are considering:

$$(7) \qquad \left\{\begin{array}{l} x_1 = y_1^{p_1}, \\ x_i = \displaystyle\sum_{q=0}^{\lambda_i} y_1^{p_i+q} \omega_{iq}(y_{i+1}, \cdots, y_n) + y_1^{p_i+\lambda_i} y_i, \\ \qquad\qquad (i = 2, \cdots, n-r), \\ x_j = y_j, \qquad\qquad (j = n-r+1, \cdots, n). \end{array}\right.$$

Here r may have any value from 0 to $n-1$. When $r=n-1$, the second group of equations is lacking in (7), and, when $r<n-1$,

$$\lambda_2 \geqq \lambda_3 \geqq \cdots \geqq \lambda_{n-r} \geqq 0.$$

The method of inversion of the transformation is obvious. After the first equation and the last r equations have been solved for y in terms of x, we can solve the last of the remainder, then the next to the last, and so on. We clearly find that the y's are functions of $(x_1^{1/p_1}, x_2, \cdots, x_n)$, meromorphic at the point $(0, 0, \cdots, 0)$.

It will be seen that the case treated by Osgood (loc. cit., "Case I") is the case where $r=n-1$. But the usefulness of the result for $r<n-1$ is in general less than when $r=n-1$, for the neighborhood of $(x)=(0)$ may be transformed by (7) into a portion of the $(y_1 \cdots y_n)$-space not lying in the neighborhood of $(y)=(0)$, (compare the transformation $x_1=y_1$, $x_2=y_1y_2$). When this is the case, the fact that the *neighborhoods* of the origins of the given $(y_1 \cdots y_n)$-space of (1) and the new $(y_1 \cdots y_n)$-space of (7) correspond is insufficient to furnish complete information regarding the inversion of (1). It may be necessary to apply these results to an infinite set of neighborhoods. Consider, for example, the transformation $x_1=y_1$, $x_2=y_1\tan y_2$.

COLUMBIA UNIVERSITY

CONCERNING LOGICAL PRINCIPLES*

BY C. H. LANGFORD

1. *Introduction.* In a recent number of this Bulletin, Dr. Alonzo Church discusses certain questions concerning the nature of the principles of logic, and advances certain views with regard to the nature and status of these principles, in particular, with regard to the status of the so-called principle of excluded middle.† It would seem that the views which Dr. Church is holding are views that are often held concerning the principles of logic, or similar to views that are often held; but it is clear that they are incompatible with certain tenets of ordinary logic that are commonly accepted; and I think it possible that those who adopt positions similar to the one Dr. Church appears to be adopting have not considered in detail the bearing of such positions on more ordinary logical conceptions, and have not assured themselves that the views they are holding are in fact compatible with other views which they would be equally inclined to accept. For this reason I wish to present, as clearly and as briefly as I can, some points concerning the nature and status of the principles of logic, and to suggest an account of these principles which is in accordance with commonly accepted tenets of ordinary logic, and which is incompatible in many respects with the interpretation of logical principles suggested by Dr. Church. In giving this account I shall be concerned on occasion to point out explicitly the bearing of the views I shall be advocating on views advanced by Dr. Church, but for the most part I shall be confined simply to presenting a different interpretation.

2. *Alternative Logics.* We may begin by considering the way in which logical principles are exemplified in the relation-

* Presented to the Society, San Francisco Section, October 29, 1927.

† *On the law of excluded middle*, this Bulletin, vol. 34 (1928), pp. 75-78.

ships in which propositions stand, one to another. We may consider, in particular, the way in which such principles are exemplified in a pair of propositions having respectively the forms $(x).fx$ and $(\exists x).\sim fx$, and we may use these expressions to denote the propositions in question. These propositions exemplify the principle of excluded middle, in that $(x).fx.\vee.(\exists x).\sim fx$ could not be false; they exemplify the principle of contradiction, in that $(x).fx.(\exists x).\sim fx$ could not be true; they exemplify logical necessitation or entailing, in that $\sim(x).fx$ entails $(\exists x).\sim fx$; they exemplify logical equivalence, in that $\sim(x).fx.\equiv.(\exists x).\sim fx$ could not be false; and, no doubt, they exemplify other logical principles and relations, which are perhaps nameless, but which we might discover and name. Now one point which is brought out by an examination of this example, and upon which I wish to place emphasis, is that the logical principles and logical relations which these propositions exemplify in the relationship in which they stand, are found there as a matter of discovery; so that the occurrence of these properties is not at all an occasion for the exercise of choice or preference on our part. Another point of equal importance, connected with the first, is that these logical properties are what may be called necessary properties, in that their occurrence is dependent upon characteristics essential to the being of the propositions themselves, and upon nothing else; but in order to bring out this point, it will be necessary to describe a distinction which is commonly known as the distinction between internal and external relations.

When we consider the relations which hold among entities of various sorts, there appears to be a fundamental division among these relations, which we can describe by saying that the occurrences of some relations are dependent solely upon intrinsic features of the terms related, whereas other relationships are fortuitous so far as the intrinsic features of the terms related are concerned; or by saying, as is often done, that some relationships are grounded in the nature of their terms and others are not. Thus, to use a simple illustration,

this pen, with which I am writing, is related to the paper upon which I am writing in a way that we can describe by saying that the pen is in contact with the paper. But it is clear that there is nothing in our conception of the terms having this relation from which the fact that they are so related could be inferred; and we can express this circumstance with regard to the relationship in which these things stand by saying that their being so related is conceptually fortuitous, or by saying that they stand in external relationship in this respect. On the other hand, there are two facts connected with this pen which are related in a way that is not conceptually fortuitous, namely, the fact that this pen is green, and the fact that it is colored. These facts are related in such a way that the first necessitates the second; and this is an internal relationship.*

Now we are often interested in making suppositions that are contrary to fact; and such suppositions are sometimes possible and sometimes not, and whether they are possible depends upon the nature of the facts in question. Thus we can make a supposition contrary to fact by supposing that this pen is not in contact with this paper, and this supposition is intelligible, and might be of interest; but when, for example, we attempt to suppose that this pen is both green and not colored, we find that our assent to this attempted supposition is merely verbal, that our words cannot retain their meanings, since being green involves being colored; and we are implicated in a species of self-contradiction, and consequent unintelligibility. Of course we can *say* that this pen is both green and not colored, and understand that what we say could not be true, that is, that there is no intelligible supposition to the effect that it is true. Now wherever internal relationships occur, there facts occur which do not admit of suppositions contrary to them; and such facts are known as analytic facts. On the other hand, wherever ex-

*The best discussion of internal and external relations with which I am acquainted is that of Professor G. E. Moore, in his *Philosophical Studies*, Chap. 9.

ternal relationships occur, there facts occur which do admit of suppositions contrary to them; and such facts are known as contingent or empirical facts.

This brings me to a point that I want to make concerning the possibility of alternatives to ordinary logical principles that shall be in some sense incompatible with these principles. It is clear that all of the facts pointed out above, with regard to the relationship in which the two propositions $(x).fx$ and $(\exists x).\sim fx$ stand, are analytic facts; and generally, it is clear, I think, that all logical facts are analytic, and thus that logical principles, which are based on logical facts, do not admit of intelligible alternatives. This means that we cannot have alternative logics; for logic is the system of all propositions expressing analytic facts of a certain kind, namely formal analytic facts, and there cannot in the nature of the case be more than one such system, actual or conceivable.

3. *Deduction.* We may now examine in some detail the nature of the connections existing between propositions, and between properties, in virtue of which we are able to argue validly from one proposition to another, or from one property to another. When we consider a pair of properties, p, q, it often happens that they are related in a way which we can describe by saying that there could not be an instance of the first that is not also an instance of the second; and when this is the case, we say that the first entails or necessitates the second, that the second is deducible from the first.* As an example of this relation, we may take a case of entailing that occurs in connection with certain properties which are similar to properties commonly used in the definition of serial order, but which differ from properties of serial order in that reference to a class K, that is, to a function fx, is omitted. It is clear that the conjunction of the property of transitivity with the contrary of the reflexive

*See Lewis, *A Survey of Symbolic Logic*, Chap. 5.

property necessitates the property "$\sim Rx_1x_2 \vee \sim Rx_2x_1$ for every x_1, x_2"; that is, that

$$(x_1, x_2, x_3) : Rx_1x_2 . Rx_2x_3 . \supset . Rx_1x_3 : . (x) . \sim Rxx$$

entails*

$$(x_1, x_2) . \sim Rx_1x_2 \vee \sim Rx_2x_1 .$$

As soon as we have envisaged clearly the meanings of these two properties, we see that the first involves the second, and that this relation is immediate and direct, that no mediating principles are required. This case is relatively simple, but in more complicated cases the situation is precisely the same; in complicated cases we often require an elaborate technique of proof, for the purpose of exhibiting to ourselves the relatedness of the properties in question, but this technique of proof is employed solely for the purpose of displaying the facts, and does not in any way condition them. Moreover, the fact that the first of the above properties necessitates the second, so far from being dependent upon some logical principle, is itself of such a kind that it can be taken as the ground of a logical principle, which we can express by saying that there could not be an instance of the propositional function

$$(x_1, x_2, x_3) : Rx_1x_2 . Rx_2x_3 . \supset . Rx_1x_3 : . (x) . \sim Rxx : . \supset . (x_1, x_2) . \sim Rx_1x_2 \vee \sim Rx_2x_1$$

that is false, just as we can express the principle of excluded middle by saying that there could not be an instance of the propositional function $p \vee \sim p$ that is false. This new principle differs from the principle of excluded middle in being more determinate; but that is a matter of degree, not an essential difference. In general, then, whatever properties $p_1, \cdots, p_n$ may be, if q is a logical consequence of $p_1 \cdots p_n$,

*It is to be noted that R is not an undefined term, but a logical function, $f(x, y)$; or perhaps it would be more accurate to say that R and similar entities, which are often called undefined terms, are simply logical functions or other items of logical structure. See Russell, *Introduction to Mathematical Philosophy*, Chap. 3.

we can say that q's being a consequence of $p_1 \cdot \cdot \cdot p_n$ does not depend upon anything outside the properties themselves, and especially that it does not depend upon anything that could be described as the logic which we assume, since the fact of q's following from $p_1 \cdot \cdot \cdot p_n$ is itself an analytic fact, namely, the fact that there could not be an instance of $p_1 \cdot \cdot \cdot p_n \cdot \sim q$ that is true.

I think it possible that the view that logical principles are in some sense premises in the deduction of one property from another, or of one proposition from another, is responsible for Dr. Church's view that we can have logics in which some ordinary logical principle is said to be not assumed, or in which some logical principle is invalid; for of course it often happens that a proposition $p_1 . p_2$ entails a proposition q, whereas p_1 alone does not entail q; and if p_2 were thought of as being some logical principle, then it might still be held that q could not be inferred if p_2 were not asserted, and that if p_1 were asserted and q denied, p_2 would have to be denied. In opposition to this view, I am maintaining that if we attempt to deny some logical principle, which is in fact valid, and on the basis of this deny certain inferences from one proposition to another, we do not get a new logic—we simply make a mistake; and similarly, that if we refuse to recognize an inference from one proposition to another, on the ground that we do not admit the logical principle which these propositions exemplify in the relationship in which they stand, then we simply make a mistake of another kind.

And I should like to introduce a remark at this point with regard to another view suggested by Dr. Church, a view to the effect that in logic we are not concerned with questions as to the truth and falsity of propositions. We do not raise questions concerning the truth and falsity of premises at the basis of abstract deductive systems, for the sufficient reason that, being properties, not propositions, these premises are neither true nor false, and we do not raise questions concerning the truth and falsity of propositions that are in-

stances of properties at the basis of abstract deductive systems, because these propositions, when they are true, are not logical truths; but we do raise questions concerning the truth of propositions which we take to be logical principles, since these propositions must be true if we are not to be mistaken in supposing that they are logical principles.

In what has been said up to this point, and especially in what has been said concerning analytic and empirical facts, and internal and external relations, there is implicit a fundamental division of properties into two classes,—the class of contingent properties on the one hand, and the class of necessary and impossible properties on the other. An ordinary property, such as $f(x, y)$, is a contingent property if and only if, in view of its structure, it could have an instance that is true, and could have an instance that is false; whereas, a property is necessary if and only if it is such that it could not have an instance that is false, and a property is impossible if and only if it is such that it could not have an instance that is true. Thus, if we form the property $f(x, y) \vee \sim f(x, y)$, we have a necessary property, which is a specific form of the property involved in the principle of excluded middle, whereas if we form the property $f(x, y) . \sim f(x, y)$, we have an impossible property, which is a specific form of the property involved in the principle of contradiction. In this connection I wish to suggest, as a possible extension of the analysis given of logical and mathematical propositions of the form "p implies q," that any proposition of logic or mathematics can be expressed as an assertion with regard to some property that that property could not have an instance that is false (and, alternatively, as an assertion with regard to a property that that property could not have an instance that is true), or as an assertion with regard to some property that that property could have an instance that is true (and, alternatively, as an assertion with regard to a property that that property could have an instance that is false,—as in theorems on non-deducibility), or as a combination of such assertions. In sets of properties

at the basis of deductive systems, and often elsewhere, contingent properties are given in isolation from the propositions into which they enter; but necessary properties do not as a rule occur in isolation. Thus, in an arithmetical proposition, say $2+3=5$, we have a necessary property whose instances express relationships of classes, and we have no conventional way of expressing this property as distinguished from the logically necessary proposition into which it enters; but if we allow $f(2, 3, 5)$ to express the arithmetical property in question, then we can, so I am holding, express the arithmetical proposition by saying that there could not be an instance of $f(2, 3, 5)$ that is false. As illustrations of contingent logical properties, we may take fx, Rxy, the cardinal number 10, the order-type ω, and the system of abstract euclidean geometry. Of course a property such as either of the last two is not in practice dealt with directly, but through a logically equivalent property, called a postulate-set or set of defining properties.

Now some properties are species of other properties, in the sense in which *green* is a species of *color*, or in which *roundness* is a species of *shape*; and this relation of species to genus is relevant to an account of the way in which propositions that are known as logical principles are related to other logical and mathematical propositions. I am holding that propositions known as logical principles do not differ in any essential respect from logical and mathematical propositions generally; but they are, as a rule, relatively simple propositions, so that their truth can be apprehended more immediately, and this is no doubt a reason for their being selected as principles. There is, however, another reason for their being selected, connected with their simplicity: the properties upon which logical principles are based, being simple, are more generic, and have many species among less generic properties, so that logical principles have many consequences among less generic propositions. Thus $(x)fx \vee (\exists x)\sim fx$ is a species of $p \vee \sim p$, and is sufficient for $p \vee \sim p$; but we do not formulate principles such as

"$(x)fx \vee (\exists x)\sim fx$ could not have an instance that is false" (or "$(\exists x)\sim fx \cdot (x)fx$ could not have an instance that is true"), because a single more generic principle will do.

It is to be noted, however, that whether we consider logical principles at all is ultimately a matter of preference, and is dependent upon whether we wish to organize logical and mathematical propositions into a deductive order—as is done, for example, in *Principia Mathematica*. One who is not interested in such organization can insist that any logical or mathematical proposition, say the proposition to the effect that π is transcendental, is just as much a logical principle as the principle of excluded middle, since the evidence for the truth of the proposition is found within the proposition itself. Of course this does not mean that relationships of deducibility which occur among logical propositions are in any sense arbitrary; it means that they are not logically prior to relationships of deducibility which occur among other propositions, and thus that it is quite unnecessary to infer the implication of one contingent proposition by another from the implication of one logical proposition by another. For when we consider the logical propositions to which the relationships of necessary properties give rise, we see that these propositions are species of ordinary logical principles, and are thus the same in kind as the logical propositions which arise from relationships of contingent properties. On the other hand, we have noted that properties which are logically impossible can be used equally well in the formulation of logical principles, and here too it is clear that the relations of an impossible property to another property—whether this other property is impossible, necessary, or contingent—simply give rise to species of ordinary principles.

We may now revert briefly to the point originally made concerning the impossibility of alternatives to logical principles, in order to connect that point with the subsequent discussion. It is clear that if the account given of logical principles is a right account, then, in view of the way in

which these principles are formulated, what precisely distinguishes a logical principle from an ordinary contingent proposition is the absence of alternative possibilities, this impossibility of alternatives being explicitly stated in the formulation of the principle itself; so that whoever holds with regard to an assigned proposition that there could be circumstances under which that proposition would fail is holding that the proposition in question is not a logical principle.

THE UNIVERSITY OF WASHINGTON

NOTE ON EINSTEIN'S EQUATION OF AN ORBIT

BY JAMES PIERPONT

In a paper* bearing the above title Morley has given an extremely elegant solution of Einstein's equation

$$\left(\frac{dx}{d\theta}\right)^2 = 2x^3 - x^2 + 2\lambda x - \lambda^2(1 - e^2), \tag{1}$$

which defines the motion of a single planet about the sun. Here r, θ are the polar coordinates of the planet, a the major semi-axis, e the eccentricity of the orbit, M the mass of the sun, and

$$x = \frac{M}{r}, \quad \lambda = \frac{M}{a(1 - e^2)}. \tag{2}$$

In Eddington units, $M = 1.45$. For Mercury, the values are

$$a = 5.8 \cdot 10^{-7}, \qquad e = 0.206, \qquad \lambda = 2.6 \cdot 10^{-8}.$$

The roots of the right side of (1) are thus, to a high degree of approximation,

$$(1 - e)\lambda, \qquad (1 + e)\lambda, \qquad \tfrac{1}{2} - 2\lambda.$$

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* American Journal of Mathematics, vol. 43 (1921), p. 29. I notice two obvious typographical errors in this paper. In the last term of (2) a should be a^2; also just below, x_1 should read $x_1 = \frac{1}{2} - 2a$.

We reduce (1) to Weierstrass' canonical form, on setting

$$x = p + \tfrac{1}{6}, \qquad d\theta = 2^{1/2}du.$$

This gives

$$\left(\frac{dp}{du}\right)^2 = 4(p - e_1)(p - e_2)(p - e_3),$$

where

(3) $e_1 = \tfrac{1}{3} - 2\lambda$, $e_2 = -\tfrac{1}{6} + \lambda(1 - e)$, $e_3 = -\tfrac{1}{6} + \lambda(1 + e)$,

the indices being chosen so that $e_2 < e_3 < e_1$.

We find now

$$\theta = 2^{1/2} \cdot t\omega_1, \qquad u = \omega_2 + t\omega_1.$$

To $t = 0$ corresponds maximum r, while to $t = 1$ corresponds minimum r. Thus as t increases from 0 to 1, the planet passes from aphelion to perihelion, and θ increases by $\Delta\theta = 2^{1/2}\omega_1$.

It only remains to calculate this quantity. Morley now remarks that "the appropriate formulas are given in works on the elliptic functions. But as the proofs of the full formulas are necessarily complicated, I shall interpolate a proof of the approximate formulas of a kind that is at once intelligible."

I wish to submit an alternate form of proof which requires only the simplest and most commonplace formulas.*

We have

$$\omega_1 = \frac{K}{(e_1 - e_2)^{1/2}}, \qquad (4), \text{ p. } 448,$$

$$k'^2 = \frac{e_1 - e_3}{e_1 - e_2}, \qquad (3), \text{ p. } 449,$$

$$q = \frac{1}{2}\,\frac{1 - k'^{1/2}}{1 + k'^{1/2}}, \qquad (3), \text{ p. } 437,$$

$$\left(\frac{2K}{\pi}\right)^{1/2} = 1 + 2q + 2q^4 + \cdots, \qquad (6), \text{ p. } 438.$$

* The numbers on the right refer to my book *Functions of a Complex Variable*. Unfortunately the most distressing confusion exists in the notations employed by different authors. For this reason I may be excused for referring to my book which contains only the briefest account of the elliptic functions. I take this occasion to correct formula (5), p. 448. The numerator in this formula should be $e_1 - e_2$ instead of $e_2 - e_1$.

Now, from (3),

$$e_1 - e_3 = \tfrac{1}{2}[1 - 2\lambda(3 + e)], \quad e_1 - e_2 = \tfrac{1}{2}[1 - 2\lambda(3 - e)],$$

$$k'^2 = [1 - 2\lambda(3 + e)][1 + 2\lambda(3 - e)] = 1 - 4\lambda e,$$

$$k'^{1/2} = 1 - \lambda e, \qquad q = \frac{1}{2}\,\frac{\lambda e}{2 - \lambda e} = \frac{1}{4}\lambda e,$$

$$\frac{2K}{\pi} = 1 + 4q, \qquad K = \pi\,\frac{1 + \lambda e}{2},$$

$$(e_1 - e_2)^{-1/2} = 2^{1/2}[1 + \lambda(3 - e)],$$

$$\omega_1 = \pi(1 + \lambda e)2^{1/2}\,\frac{1 + \lambda(3 - e)}{2} = \pi\,\frac{1 + 3\lambda}{2^{1/2}}.$$

Thus $\Delta\theta = \pi + 3\pi\lambda$. Hence the advance in the radius vector from perihelion to perihelion over 360° is $\phi = 6\pi\lambda$. Now in the Newtonian theory of elliptic motion $M = 4\pi^2 a^3/T^2$. Substituting in (2) and using one second as the unit of time, we find

$$\phi = \frac{24\pi^3 a^2}{c^2(1 - e^2)T^2}.$$

This is Einstein's celebrated formula for the advance of the perihelion of a planet.

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ON LEVI-CIVITA'S THEORY OF PARALLELISM

BY C. E. WEATHERBURN

1. *Parallel Displacements.* The use of two-parametric differential invariants* (grad, div etc.) for a surface not only leads very neatly and quickly to the condition of parallelism and the known theorems, but also puts in evidence other results hitherto unnoticed. In particular, some interesting properties of Tchebychef nets are discovered.

Consider a curve C drawn on a surface S in ordinary space. Let $\boldsymbol{R}$ be a vector of constant length (which may be taken as unity), and a point-function along C, being everywhere tangential to the surface. Then $\boldsymbol{R}$ is said to undergo a parallel displacement along C, or to remain parallel to itself for displacement along this curve, provided the derivative of $\boldsymbol{R}$ along C is everywhere normal to the surface.† We lose no generality by regarding C as a member of a family T of curves, with unit tangent $\boldsymbol{t}$ and arc-length s, and $\boldsymbol{R}$ as the unit tangent to a second family cutting the former at a variable angle θ, so that θ is the angle of rotation from $\boldsymbol{t}$ to $\boldsymbol{R}$ in the positive sense about the unit normal $\boldsymbol{n}$. If then $\boldsymbol{m}$ is the unit surface vector $\boldsymbol{n}\times\boldsymbol{t}$, perpendicular to $\boldsymbol{t}$, we may write

$$\boldsymbol{R} = \boldsymbol{t}\cos\theta + \boldsymbol{m}\sin\theta.$$

Let a prime denote differentiation with respect to s, that is to say, in the direction of $\boldsymbol{t}$. Then in order that $\boldsymbol{R}$ may be normal to S we must have

$$0 = \boldsymbol{t}\cdot\boldsymbol{R}' = \sin\theta\left[\boldsymbol{t}'\boldsymbol{m}\cdot - \theta'\right],$$

and

$$0 = \boldsymbol{m}\cdot\boldsymbol{R}' = \cos\theta\left[\boldsymbol{m}\cdot\boldsymbol{t}' + \theta'\right],$$

* All the differential invariants of this article are two-parametric invariants. See the writer's *Differential Geometry*, Chap. 12, Cambridge University Press, or Quarterly Journal of Mathematics, vol. 50, pp. 230–269.

† See Levi-Civita, *The Absolute Differential Calculus*, pp. 101–119.

from either of which it follows that

$$\frac{d\theta}{ds} = \operatorname{div} \boldsymbol{m} = -\gamma \tag{1}$$

where γ denotes* the geodesic curvature of C. This is Levi-Civita's formula† for the arc-rate of increase of θ along C. From it follow the theorems of the constancy of the angle between two vectors undergoing parallel displacements along C, and of the constancy of θ for parallel displacement along a geodesic.

Since the derivative of $\boldsymbol{R}$ along C is normal to S, its *magnitude* is given by

$$\boldsymbol{n}\cdot\boldsymbol{R}' = \boldsymbol{n}\cdot(\boldsymbol{t}'\cos\theta + \boldsymbol{m}'\sin\theta) = \kappa_n\cos\theta + \tau\sin\theta, \tag{2}$$

where κ_n is the normal curvature of the surface in the direction of $\boldsymbol{t}$, and τ is the torsion of the geodesic tangent to C, or the moment‡ of the family of curves T. This magnitude is clearly the resolved part of the vector $\kappa_n\boldsymbol{t}+\tau\boldsymbol{m}$ in the direction of $\boldsymbol{R}$.

Again, consider a family of curves A with unit tangent $\boldsymbol{a}$, and their orthogonal trajectories B with unit tangent $\boldsymbol{b}$, so that $\boldsymbol{n}=\boldsymbol{a}\times\boldsymbol{b}$. Let a third family T cut the first at a variable angle $-\theta$, so that the unit tangent $\boldsymbol{t}$ to this family is given by $\boldsymbol{t} = \boldsymbol{a}\cos\theta - \boldsymbol{b}\sin\theta$. Then the geodesic curvature γ of the last family is found from§

$$\begin{aligned} -\gamma &= \operatorname{div}(\boldsymbol{a}\sin\theta + \boldsymbol{b}\cos\theta) \\ &= \sin\theta\operatorname{div}\boldsymbol{a} + \cos\theta\operatorname{div}\boldsymbol{b} + \boldsymbol{t}\cdot\nabla\theta, \end{aligned} \tag{3}$$

where $\nabla\theta$ denotes the gradient of θ. If now the vector $\boldsymbol{a}$ is such that it suffers a parallel displacement along a curve of the family T, we have $\boldsymbol{t}\cdot\nabla\theta = -\gamma$, and (3) becomes

$$\sin\theta\operatorname{div}\boldsymbol{a} + \cos\theta\operatorname{div}\boldsymbol{b} = 0, \tag{3'}$$

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* See the writer's *Differential Geometry*, p. 234.

† Levi-Civita, Palermo Rendiconti, vol. 42 (1917), p. 185.

‡ *Differential Geometry*, §130.

§ Ibid., §121.

from which two theorems may be deduced. First suppose that the curves of the family A constitute a family of parallels. Then div $\boldsymbol{a}$ vanishes* identically; and the last equation shows that either $\cos\theta = 0$ or div $\boldsymbol{b} = 0$. The second of these alternatives can be satisfied only where the geodesic curvature of the parallel curves is zero, that is to say, on their line of normal curvature. Hence we have the following theorem.

THEOREM 1. *If the tangent to a family of parallel curves undergoes a parallel displacement along a transversal, either this transversal is an orthogonal trajectory (and therefore a geodesic), or else it is a line of normal curvature of the family of parallels.*

Secondly, let the family A be one of geodesics. Then div $\boldsymbol{b}$ vanishes identically, and the previous equation (3′) takes the form $\sin\theta \operatorname{div} \mathbf{a} = 0$. Since then $\sin\theta$ is not zero, this requires div $\boldsymbol{a} = 0$, which holds only at the line of striction† of the family of geodesics. Hence we have Bianchi's theorem‡ that if the tangent to a family of geodesics on a surface undergoes a parallel displacement along a transversal this transversal is a line of striction of the family of geodesics.

From (3) follows also another result which, though not connected with the parallelism of Levi-Civita, may be mentioned here and will be used in §2. Suppose that the two families T and A are geodesics, and that the angle θ is constant along each member of the former. Then γ, div $\boldsymbol{b}$ and $\boldsymbol{t}\cdot\nabla\theta$ all vanish identically, and therefore by (3), so also does div $\boldsymbol{a}$. The family A is thus also a family of parallels; and the surface is developable. For the second (or Gaussian) curvature K is given by§

$$(4) \qquad K = -(\operatorname{div}\boldsymbol{a} \operatorname{div}\boldsymbol{a} + \boldsymbol{b} \operatorname{div}\boldsymbol{b}) = 0.$$

* *Differential Geometry*, p. 258.
† Ibid., §126.
‡ *Geometria Differenziale*, vol. 2, p. 802.
§ *Differential Geometry*, §125.

Hence we have the following theorem:

THEOREM 2. *If a family of geodesics cut a family of geodesics (or parallels) at an oblique angle which is constant along each member of the former, the latter are both geodesics and parallels, and the surface is developable.*

2. *Tchebychef Nets.* A Tchebychef system (or net) on a given surface consists of two families of curves, U, V, such that the tangent to each family undergoes a parallel displacement along each curve of the other family.* If u, v, are the unit tangents to the two families U, V, respectively, the derivative of v in the direction of u is $u \cdot \nabla v$; and since this is normal to the surface we must have $u \cdot \nabla v \cdot u = 0$. Similarly it follows that $v \cdot \Delta u \cdot v = 0$. Conversely, if these relations hold, both derivatives are normal to the surface, and we have the following theorem.

THEOREM 3. *Necessary and sufficient conditions that the two families of curves with unit tangents u, v constitute a Tchebychef system are expressed by the equations*

$$u \cdot \nabla v \cdot u = 0, \qquad v \cdot \nabla u \cdot v = 0. \tag{5}$$

Now the tendency of a family of curves in any direction is the resolved part in that direction of the derivative of the unit tangent in that direction.† The equations (5) therefore express that, at any point of the surface, the tendency of each family in the direction of the other is zero. If then we introduce the conics‡

$$r \cdot \nabla u \cdot r = 1, \tag{6}$$

and

$$r \cdot \nabla v \cdot r = 1, \tag{7}$$

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* Bianchi, loc. cit., p. 809.

† See §1 of a paper by the author, *On some properties of a family of curves on a surface*, recently communicated to the Edinburgh Mathematical Society.

‡ For this form of the equation of a quadric see the author's *Advanced Vector Analysis*, G. Bell and Sons, §66.

whose centers are at P, the origin of the vector $\boldsymbol{r}$, and whose plane is the tangent plane at P, it follows from (5) that $\boldsymbol{v}$ has the direction of one of the asymptotes of (6). But the other asymptote has the direction of $\boldsymbol{u}$; so that the tangents to the two families of curves are the asymptotes of (6). The same is true of the conic (7), and the two conics are therefore similar. The tendency of the family U in any direction varies inversely as the square of the radius of the conic (6) in that direction.* Thus the ratio of the tendencies of the two families in any direction at P is invariant. But the sum of the tendencies of a family in two directions at right angles is equal to the divergence of the family. Hence we may state the following theorem.

THEOREM 4. *The ratio of the tendencies of the two families of a Tchebychef system in any direction at a given point is invariant, and equal to the ratio of their divergences at that point.*

Also, at points where the two families U and V cut orthogonally the asymptotes of the conics (6) and (7) are at right angles. The sum of the tendencies in two perpendicular directions is zero for each family, so that the divergence of each is zero. Hence we have the following theorem.

THEOREM 5. *The locus of the points at which the two families of a Tchebychef system cut orthogonally is a common line of striction for the two families.*

Let $\boldsymbol{p}$, $\boldsymbol{q}$ denote the unit tangential vectors $\boldsymbol{n}\times\boldsymbol{u}$, $\boldsymbol{n}\times\boldsymbol{v}$ perpendicular to $\boldsymbol{u}$, $\boldsymbol{v}$ respectively. Then the geodesic curvatures γ_1, γ_2 of the two curves U, V are given by the equations

$$-\gamma_1 = \operatorname{div}\boldsymbol{p}, \quad -\gamma_2 = \operatorname{div}\boldsymbol{q}.$$

Let θ be the angle of rotation from $\boldsymbol{u}$ to $\boldsymbol{v}$ in the positive sense. Then, since the vector $\boldsymbol{u}$ undergoes a parallel displacement along the curve V, it follows from (3) and (3′) that

$$\operatorname{div}\boldsymbol{u}\tan\theta = \operatorname{div}\boldsymbol{p} = -\gamma_1. \tag{8}$$

* Edinburgh Mathematical Society, loc. cit., §1.

From equation (8), $\gamma_1=0$ implies div $\boldsymbol{u}=0$ since tan $\theta\neq 0$. We may therefore state the following theorem.

THEOREM 6. *A line of normal curvature of either family of a Tchebychef system is also a line of striction of that family.*

The *gradient* of θ may be expressed in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$ as follows. Taking the rot of both members of the identity

$$\boldsymbol{u}\times\boldsymbol{v}=\boldsymbol{n}\sin\theta, \tag{9}$$

we have*

$$\boldsymbol{v}\cdot\nabla\boldsymbol{u}-\boldsymbol{u}\cdot\nabla\boldsymbol{v}+\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{v}-\boldsymbol{v}\,\mathrm{div}\,\boldsymbol{u}=\cos\theta\nabla\theta\times\boldsymbol{n}. \tag{10}$$

The derivatives represented by the first two terms are parallel to $\boldsymbol{n}$. Therefore, on forming the vector product of both sides with $\boldsymbol{n}$, we have*

$$\cos\theta\nabla\theta=\boldsymbol{n}\times(\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{v}-\boldsymbol{v}\,\mathrm{div}\,\boldsymbol{u}). \tag{11}$$

Since $\cos\theta=\boldsymbol{u}\cdot\boldsymbol{v}$, this formula expresses $\nabla\theta$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$. Where $\cos\theta$ is zero, both div $\boldsymbol{u}$ and div $\boldsymbol{v}$ must vanish, and we see again that this is a common line of striction of the two families U and V.

If the two families of a Tchebychef net cut at a constant angle, $\nabla\theta$ is everywhere zero, so that div $\boldsymbol{u}$ and div $\boldsymbol{v}$ must vanish identically. Thus both families are families of parallel curves; and, as they cut at a constant angle, it follows from the last theorem of §1 that they are also geodesics, and the surface is developable. Hence we have the following theorem.

THEOREM 7. *If the two families of a Tchebychef net cut at a constant angle, each is a family of parallel geodesics, and the surface is developable.*

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* *Differential Geometry*, p. 237, Ex. 10.

PLANE NETS WHOSE FIRST AND MINUS FIRST LAPLACIAN TRANSFORMS EACH DEGENERATE INTO A STRAIGHT LINE*

BY J. O. HASSLER

1. *Introduction.* In the projective theory of plane nets, as developed by Wilczynski,† the members $y_1(u, v)$, $y_2(u, v)$, $y_3(u, v)$ of any fundamental system of solutions of a completely integrable system of partial differential equations of the form

$$(1) \qquad \begin{cases} y_{uu} = ay_u + by_v + cy, \\ y_{uv} = a'y_u + b'y_v + c'y, \\ y_{vv} = a''y_u + b''y_v + c''y \end{cases}$$

are interpreted as the homogeneous coordinates of a point P_y in a plane, defining a non-degenerate net of plane curves consisting of two one-parameter families. The invariants and covariants of the system (1) under the transformations

$$(2) \qquad y = \lambda(u,v)\bar{y},$$

and

$$(3) \qquad \bar{u} = U(u), \text{ and } \bar{v} = V(v)$$

may be interpreted geometrically by certain projective properties of the net. The two covariants

$$(4) \qquad \rho = y_u - b'y, \quad \sigma = y_v - a'y$$

may be taken as the homogeneous coordinates of two points P_ρ and P_σ when we substitute successively for y the values y_1, y_2, and y_3. As u and v vary P_σ and P_ρ describe nets called the first and minus first Laplacian transforms of the original net.

* Presented to the Society, April 6, 1928.

† E. J. Wilczynski, *One-parameter families and nets of plane curves*, Transactions of this Society, vol. 12 (1911), pp. 473–510.

In this paper we consider nets such that the curves $u=$const. of the first Laplacian transform all degenerate into one straight line while the curves $v=$const. go into points on the line, and the curves $v=$const. of the minus first Laplacian transformd egenerate into one straight line while the curves $u=$const. go into points on the line. The invariants and covariants of the net are computed and the net determined by certain boundary conditions.

2. *The Canonical Form of the Equations and the Invariants.* If we apply the transformation (2) to the system of equations (1), we obtain

$$(1') \qquad \begin{cases} y_{uu} = -B'y_u + By_v + Cy, \\ y_{uv} = A'y_u + B'y_v + C'y, \\ y_{vv} = A''y_u - A'y_v + C''y, \end{cases}$$

where the coefficients B', B, C, etc., are functions of a, b, c, a', b', c', a'', b'', c'', λ and their derivatives.* The system is then said to be in its canonical form. The coefficients B and A'' are relative invariants of the system under the transformations (2) and (3). The five remaining fundamental invariants are

$$(5) \qquad \begin{aligned} \mathfrak{A}' &= A' + \frac{A_v''}{6A''}, \quad \mathfrak{B}' = B' + \frac{B_u}{6B}, \quad \mathfrak{C}' = C' + A'B', \\ \mathfrak{C} &= C - B_u' - 2B'^2 + A'B, \\ \mathfrak{C}'' &= C'' - A_v' - 2A'^2 + A''B'. \end{aligned}$$

Two other invariants,†

$$(6) \qquad H = C' + A'B' - A_u', \quad K = C' + A'B' - B_v',$$

have a special significance in connection with the Laplacian transforms.

Wilczynski has shown‡ that if $H=0$, the curves $v=$const. of the first Laplacian transform degenerate into points and

* Wilczynski, loc. cit., equations (13).

† Wilczynski, loc. cit., equations (21) and (29).

‡ Wilczynski, loc. cit., pp. 489–490.

the curves $u=$const. all coincide with the locus of these points, making the degenerate transformed net consist of a single curve. Similarly, if $K=0$, the minus first Laplacian transform becomes a single curve $v=$const. Furthermore, if*

$$A''^2K + A''\mathfrak{C}_v' - \mathfrak{C}''A_v'' = 0, \tag{7}$$

the curves $u=$const. of the first Laplacian transform become straight lines, and if

$$B^2H + B\mathfrak{C}_u - \mathfrak{C}B_u = 0, \tag{8}$$

the curves $v=$const. of the minus first Laplacian transform become straight lines. We shall consider the case where equations (7) and (8) are satisfied simultaneously with

$$H = K = 0. \tag{9}$$

From (7) and (8), by means of (9), we obtain, after integrating,

$$\log\frac{\mathfrak{C}''}{A''} = \phi(u), \qquad \log\frac{\mathfrak{C}}{B} = \psi(v),$$

where ϕ and ψ are arbitrary functions of u and v, respectively. The quotients $\mathfrak{C}''/A''$ and $\mathfrak{C}/B$ are relative invariants. If we apply the transformation (3), they are transformed into

$$\frac{\bar{\mathfrak{C}}''}{\bar{A}''} = \frac{1}{U'}\frac{\mathfrak{C}''}{A''}, \qquad \frac{\mathfrak{C}}{\bar{B}} = \frac{1}{V'}\frac{\mathfrak{C}}{B}.$$

If we choose U and V so that $U'=e^{-\phi(u)}$ and $V'=e^{-\psi(v)}$, we have

$$\mathfrak{C}'' = A'', \qquad \mathfrak{C} = B. \tag{10}$$

The system (1) is subject to certain integrability conditions† which, if used with (5), (6), (9), and (10), enable us to express all of the coefficients of the canonical form in terms of B and A''. In fact, we find that

* Wilczynski, loc. cit., equation (54a).

† Wilczynski, loc. cit., equations (5) and (14).

$$(11)\quad \begin{cases} B = B, \quad C = \dfrac{2}{9}(1 + B + B^2) - \dfrac{1}{3}(B_u + B_v), \\ A' = \dfrac{1-A''}{3}, \quad B' = \dfrac{1-B}{3}, \quad C' = \dfrac{1}{9}(2A''B + A'' + B - 1), \\ A'' = A'', \quad C'' = \dfrac{2}{9}(1 + A'' + A''^2) - \dfrac{1}{3}(A_u'' + A_v''), \end{cases}$$

with the further conditions that the invariants A'' and B must satisfy the partial differential equations

$$(12)\qquad A_u'' = B_v = -A''B.$$

The general solution of (12) is*

$$(13)\qquad A'' = \frac{\psi'(v)}{\phi(u) + \psi(v)}, \quad B = \frac{\phi'(u)}{\phi(u) + \psi(v)},$$

where ϕ and ψ are arbitrary functions of u and v, respectively. It is easy to verify that the integrability conditions are identically satisfied if a system of partial differential equations of the form (1′) has coefficients defined by (11) and (13). Hence we may state the following conclusion.

If the coefficients of a system of partial differential equations of the form (1′) *satisfy conditions* (11) *and* (13), *any fundamental system of solutions* y_1, y_2, y_3 *of* (1′) *defines a net whose first and minus first Laplacian transforms each degenerate into a straight line. Conversely, any net whose first and minus first Laplacian transforms each degenerate into a straight line gives rise to a system of partial differential equations of the form* (1′) *whose coefficients satisfy conditions* (11) *and* (13).

3. *Determination of the Net by Boundary Conditions.* In the projective differential geometry of plane curves† three linearly independent solutions $y_1(x)$, $y_2(x)$, $y_3(x)$, of a linear homogeneous differential equation of the third order,

* Acknowledgement is hereby given to G. E. Raynor, of the University of Oklahoma, for the solution of these equations.

† E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, Leipzig, B. G. Teubner, 1916, pp. 58–61.

$$\frac{d^3y}{dx^3} + 3p_1\frac{d^2y}{dx^2} + 3p_2\frac{dy}{dx} + p_3y = 0, \tag{14}$$

are interpreted as the homogeneous coordinates of a point on a plane.

Two functions of the coefficients of (14), called θ_3 and θ_8 by Wilczynski, which remain invariant under the transformations

$$y = \lambda(x)\eta, \qquad x = f(\xi),$$

are sufficient to determine the projective properties of the curve. If, then, two arbitrary functions $\omega_3(x)$ and $\omega_8(x)$ be given there exists a curve whose invariants θ_3 and θ_8 are respectively equal to these given functions and this curve will be uniquely determined except for projective transformations. In particular, if $\theta_3 = 0$, the corresponding curve is a conic.

If we differentiate both members of the first equation of (1′) with respect to u and then eliminate y_v and y_{uv}, we obtain an equation of the form (14). We may then compute the invariants θ_3 and θ_8 for a curve $v = \text{const.}$ of the net under consideration. In fact, if we employ the equations derived by Wilczynski, making use of equations (11) and (13), we find

$$\begin{aligned} 54\theta_3 &= -9\frac{\phi^{iv}}{\phi'} + 9\frac{\phi'''}{\phi'} + 6\frac{\phi''}{\phi'} + 45\frac{\phi''\phi'''}{(\phi')^2} \\ &\qquad - 15\frac{\phi''^2}{\phi'^2} - 40\frac{\phi''^2}{\phi'^2} + 4, \\ \theta_8 &= 6\theta_3\theta_3'' - 7\theta_3'^2 - 27P_2\theta_3^2, \end{aligned} \tag{15}$$

where the accents indicate derivatives with respect to u and where

$$P_2 = p_2 - p_1^2 - p_1' = \frac{\phi'''}{3\phi'} - 4\frac{\phi''^2}{9\phi'^2} - \frac{\phi''}{9\phi'} - \frac{1}{9}.$$

We notice that the invariants are independent of v, since ϕ is a function of u alone. All the curves $v = \text{const.}$ are projectively equivalent.

Let C be any analytic curve, not a straight line, and let equation (14) be its differential equation with respect to any independent variable x to which it may be referred parametrically. Let $\theta_3=\omega_3(x)$ and $\theta_8=\omega_8(x)$ be the values of its invariants as functions of x. If the curve $v=v_0$ of the net coincides with C it must be possible to determine u as a function of x so that the equations

$$\left\{\begin{aligned}\theta_3(u) &= \omega_3(x),\\ \theta_8(u) &= \omega_8(x)\end{aligned}\right. \tag{16}$$

are identically satisfied. On the other hand, if equations (16) are identically satisfied the curve C will be projectively equivalent to the curve $v=v_0$ of the net.

To reduce the order of the differential equations we make the substitution

$$w=\frac{\phi''}{\phi'} \tag{17}$$

in equations (15). If we substitute the values of θ_3 and θ_8 from (15) in (16) we find that the differential equations to be satisfied by w and u as functions of x have the form

$$\left\{\begin{aligned}&-9\frac{d^2w}{du^2}+9(1+2w)\frac{dw}{du}-4w^3-6w^2\\ &\qquad\qquad +6w+4=54\omega_3(x),\\ &6\omega_3\frac{d^2\omega_3}{du^2}-7\left(\frac{d\omega_3}{du}\right)^2-3\left(\frac{dw}{du}-w^2-w-1\right)\omega_3^2=\omega_8(x).\end{aligned}\right. \tag{18}$$

If C be not a conic, so that $\omega_3(x)\neq 0$, we can by a suitably chosen transformation of the independent variable x make $\omega_3=1$.* We may then, after using x as the independent variable in all the differentiation, write the equations in the form

* Wilczynski, *Projective Differential Geometry*, p. 61.

$$(19)\quad \begin{cases} -9\dfrac{d}{dx}\left[\dfrac{\dfrac{dw}{dx}}{\dfrac{du}{dx}}\right]\dfrac{1}{\dfrac{du}{dx}} + 9(1+2w)\dfrac{\dfrac{dw}{dx}}{\dfrac{du}{dx}} - 4w^2 \\ \qquad\qquad - 6w^2 + 6w - 50 = 0, \\ -3\dfrac{\dfrac{dw}{dx}}{\dfrac{du}{dx}} + 3w^2 + 3w + 3 = \omega_3(x). \end{cases}$$

If we use the second equation to simplify the first, we obtain the form

$$(20)\quad \begin{cases} -9(1+2w)\dfrac{dw}{dx} + (14w^3 + 21w^2 + 33w - 41 \\ -6w\,\omega_3(x) - 3\omega_3(x))\dfrac{du}{dx} + 3\omega_3' = 0, \\ 3\dfrac{dw}{dx} - (3w^2 + 3w + 3 - \omega_3(x))\dfrac{du}{dx} = 0. \end{cases}$$

We have in (20) two differential equations of the first order with two dependent variables w and u and one independent variable x. There exist then two integrals involving the variables and two arbitrary constants.

From (17), ϕ can be determined by two quadratures, which brings the total number of arbitrary constants up to four. The variable u enters only through its derivative du/dx. Hence $u+k$, where k is an arbitrary constant, will satisfy the same system as u. By fixing arbitrarily the origin of the u-scale k may be made equal to zero. Thus we determine ϕ as a function of u except for three arbitrary constants.

If C be a conic, $\omega_3(x) = 0$, and equations (18) reduce to the single equation

$$(21)\quad -9\frac{d^2w}{du^2} + 9(1+2w)\frac{dw}{du} - 4w^3 - 6w^2 + 6w + 4 = 0,$$

for which there exists an integral involving w as a function of u and two arbitrary constants. As in the preceding case ϕ may be determined as a function of u except for three arbitrary constants.

If we differentiate the third equation of (1′) with respect to v and eliminate y_u and y_{uv}, we obtain an equation in the form (14) where v is the independent variable. By means of this we study the projective properties of the curves $u=$const. and by choosing an arbitrary curve C' we can determine the function $\psi(v)$ except for three arbitrary constants.

Equations (13) show that the invariants A'' and B are determined except for six arbitrary constants. We shall now proceed to determine these six constants by choosing the straight lines of the Laplacian transforms.

Let $f(y_1, y_2, y_3)=0$ be the equation of the curve $C(v=v_0)$, where it is to be understood that y_1, y_2, and y_3 form a fundamental system of solutions of equations (1). Let $F(\rho_1, \rho_2, \rho_3)=0$ be the equation of the corresponding curve C_{-1}, generated by the point P_ρ, of the minus first Laplacian transform, where each ρ is defined as a function of y by the first of equations (4) which, on account of (11), become

$$(4') \qquad \rho = y_u - \frac{1-B}{3}y, \quad \sigma = y_v - \frac{1-A''}{3}y.$$

$F(\rho_1, \rho_2, \rho_3)$ becomes a function of y_1, y_2, y_3, and B. According to the conditions set forth in this paper C_{-1} is a straight line, the degenerate minus first Laplacian transform. If we choose this line arbitrarily we will have two relations between the arbitrary constants in B.

Furthermore, equations (4′) show that the point P_ρ lies on the tangent to the curve C at P_y and the point P_σ on the tangent to C'. If we choose a fixed point $u=u_0$ on C the point P_ρ can occupy only a single infinity of positions. Since the line C_{-1} has been prescribed, the position of P_ρ will be determined, thus giving us a third relation between the arbitrary constants.

In a similar manner we may secure three relations between the constants by choosing a straight line C_1 to be the degenerate first Laplacian transform and a point $v=v_0$ on the curve C'. The six relations are sufficient to determine the six arbitrary constants. We may therefore state the following conclusion.

Choose two non-rectilinear but otherwise arbitrary analytic curves C and C' intersecting in a point P and having distinct tangents T and T' at P. Choose an arbitrary straight line C_{-1} intersecting T and another line C_1 intersecting T'. There exists one and only one net which contains the curves C and C' and which moreover has C_1 and C_{-1} for its degenerate first and minus first Laplacian transforms.

The family of curves $v=$const. may be obtained from the curve C by projective transformations. Similarly, the family $u=$const. may be obtained from C'.

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SOME PROPERTIES OF UPPER SEMI-CONTINUOUS COLLECTIONS OF BOUNDED CONTINUA*

BY W. A. WILSON

1. *Introduction.* If $T=\{t\}$ denotes a closed set of points and with each point t there is associated a unique bounded continuum X (or X_t) in such a way that (a) $X_t \cdot X_{t'}=0$ if $t\neq t'$, (b) at each point $t=\tau$ of T the upper closed limit of X_t as $t\to\tau$ is a part of X_τ, we say that $X=f(t)$ is an upper semi-continuous function in T. The collection of continua $\{X\}$ is also known as an upper semi-continuous collection of continua. These aggregates have been discussed by various writers here and abroad and enjoy numerous interesting properties.

R. L. Moore, in particular, has given an extensive treat-

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* Presented to the Society, February 25, 1928.

ment of the subject† and among other things has shown that, if the continua $\{X\}$ all lie in a plane but none of them cuts the plane, then the complement of $M=\Sigma[X]$ is a simply connected region whose frontier is a part of M if T is a simple arc and the complement of M is two simply connected regions whose frontiers are parts of M if T is a circumference. It is easily seen by examples that the frontiers of the complementary regions need not coincide with M in either case; it is the purpose of this note to give the conditions under which they do.‡

To this end we define $X=f(t)$ as a *minimal* upper semi-continuous function in T if there exists no upper semi-continuous function $Y=g(t)$ such that at every point t, $Y \subset X$, and at some point $Y \neq X$. If T denotes the interval $(-1 \leqq t \leqq 1)$ and in some plane we let X_t be the point $(t, \sin(1/t))$ when $t \neq 0$ and the point set $x=0$, $-1 \leqq y \leqq 2$ when $t=0$, then $X=f(t)$ is an upper semi-continuous function which is not a minimal function, but becomes so if we replace $X_0=f(0)$ by the set $x=0$, $-1 \leqq y \leqq 1$. An example of a minimal upper semi-continuous function where no X is a point is given by the author in this Bulletin, vol. 32, p. 679.

2. *Notation.* The following notation will be convenient. If $X=f(t)$ in T and $M=\Sigma[X]$ we write $M=F(T)$. If T is a bounded continuum and $f(t)$ is upper semi-continuous, it is obvious that M is a bounded continuum; in this case we say that X is an *element* of M.

If T is a simple arc ab, $M=F(ab)$ will be called a generalized arc, or simply an arc if no confusion is caused. This may be denoted by X_aX_b and the elements X_a and X_b will be called the ends. Likewise, $M-(X_a+X_b)$ is

† R. L. Moore, *Concerning upper semi-continuous collections of continua*, Transactions of this Society, vol. 27, pp. 416–428.

‡ The attention of the reader is directed to an article by C. Kuratowski, *Sur la structure des frontières communes à deux régions*, Fundamenta Mathematicae, vol. 12 (1928), pp. 20–42, of which an advance copy was received while this paper was in press. Although Kuratowski's article is concerned chiefly with the converse problem, the reader will note a certain degree of similarity between the two papers.

called a (*generalized*) *open arc* and denoted by $X_a^* X_b^*$.

If T is a circumference C, $M = F(C)$ is called a (*generalized*) *simple closed curve*. Obviously any two elements X_1 and X_2 divide M into two arcs having X_1 and X_2 as end elements and no other common points.

The plane will be denoted throughout by Z.

3. *Certain Corollaries.* Certain properties of the sets under consideration are either corollaries of Moore's work or are so easily demonstrated that their proofs are omitted.

(a) *If $M = F(ab)$ is a generalized arc in a plane Z, no sub-continuum of M separates X_a from X_b unless some element of M does.*

(b) *If $M = F(C)$ is a generalized simple closed curve in a plane, no two elements of M are separated by a sub-continuum of M.*

These are readily proved with the aid of a theorem of Janiszewski.†

(c) *In a plane Z let $M = F(ab)$ be a generalized arc and no element of M separate X_a from X_b, or let $M = F(C)$ be a generalized simple closed curve. Then there is not more than one element X such that a bounded component of $Z - X$ contains $M - X$.*

(d) *In a plane Z let $M = F(ab)$ be a generalized arc and no element of M separate X_a from X_b, or let $M = F(C)$ be a generalized simple closed curve. For each element X let the component of $Z - X$ containing $M - X$ be unbounded. Let Y be the union of X and the components of $Z - X$ containing no points of M. Then $Y = g(t)$ is upper semi-continuous in ab or C, respectively.*

4. LEMMA. *Let $M = F(C)$ be a generalized simple closed curve in a plane Z. Then $Z - M$ has two components whose frontiers have points in every element of M and every other component has a frontier which is a part of some element.*

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† Z. Janiszewski, *Sur les coupures du plan faites par des continus*, Prace Matematyczno-Fizyczne, vol. 26, Theorem A. See also R. L. Moore, *Concerning the prime parts of certain continua which separate the plane*, Proceedings of the National Academy of Sciences, vol. 10, p. 173.

PROOF. This is really a corollary of R. L. Moore's work. By §3 (c) there is at most one element X such that a bounded component of $Z-X$ contains $M-X$. Since inversion with respect to a point within this component will make its image unbounded, there is no loss in generality if we assume that for every element X the unbounded component of $Z-X$ contains $M-X$.

Defining $Y=g(t)$ as in §3 (d), it follows from this reference that $N=\Sigma[Y]$ is a generalized simple closed curve no element of which separates Z. In this case Moore has shown (loc. cit., Theorem 11) that $Z-N$ consists of two components R and S, and the frontier of each of these has at least one point on every element Y. It is readily seen that these are also components of $Z-M$. Hence the first part of the lemma is proved.

That the frontier of each of the other components of $Z-M$ is a part of some element X is a consequence of the definition of $Y=g(t)$.

DEFINITION. The components of $Z-M$ which have frontier points on every element of M will be called *principal* components.

COROLLARY. *Let $C=\{t\}$ be a circumference, let $X=f(t)$ be an upper semi-continuous function defined over C, and let $M=\Sigma[X]$ lie in the plane Z. If M is the common frontier of two components of $Z-M$, then $f(t)$ is a minimal upper semi-continuous function.*

PROOF. Let R and S be the components of $Z-M$ having the frontier M. If the theorem is not true, let $Y=g(t)$ be upper semi-continuous over C, let $Y \subset X$ for every t, and let $Y \neq X$ for some t. Let $N=\Sigma[Y]$.

By the above lemma, $Z-N$ has two principal components R' and S'. Since M is an irreducible cut of Z between points of R and S and N is a proper part of M, R and S lie in the same component of $Z-N$, and this must contain all the points of $M-N$. Suppose that $S' \cdot (R+S)=0$. Then the frontier of S' is a part of some element of M, as S' is a com-

ponent of $Z-M$. This is a contradiction, since N contains points of every element of M.

5. LEMMA. *Let $M=F(C)$ be a generalized simple closed curve in a plane Z, and let R be one of the principal components of $Z-M$. Let a and b be points of M accessible from R and lying on different elements A and B of M. Let A and B divide M into the arcs M_1 and M_2. If F is the frontier of R, $F=H+K$, where H and K are sub-continua of M_1 and M_2, respectively, joining a and b.*

PROOF. Let m be a point of R and ma and mb be simple arcs lying in R except for the points a and b and having only m in common. R. L. Moore has shown (loc. cit., p. 423) that the arc $ab=ma+mb$ divides R into two simply connected regions R_1 and R_2 such that their frontiers are parts of $M_1+a^*b^*$ and $M_2+a^*b^*$, respectively.

Let H and K, respectively, denote those points of these frontiers not on a^*b^*; then $H \subset M_1$, $K \subset M_2$. Since the frontier of R_1 is a continuum, every point of H can be joined to a or b by a sub-continuum of H. If H is not a continuum, $H=H_1+H_2$, where H_1 and H_2 are continua containing a and b, respectively, and $H_1 \cdot H_2=0$. This hypothesis would give a contradiction by the theorem of Janiszewski referred to earlier, for neither ab, H_1, nor H_2 separates R_1 from R_2, while $(ab)\cdot H_1=a$, $(ab)\cdot H_2=b$, and $(ab+H_1)\cdot(ab+H_2)=ab$. Thus H, and in like manner K, is a continuum.

Now $F \supset H+K$. On the other hand every frontier point of R is necessarily one of R_1, or of R_2, or of both. Hence $F=H+K$.

6. THEOREM. *Let $C=\{t\}$ be a circumference, let $X=f(t)$ be a minimal upper semi-continuous function defined over C, and let $M=\Sigma[X]$ lie in the plane Z. Then M is the frontier of two components of $Z-M$ and the frontier of each of the remaining components is a part of some element of M.*

PROOF. The last assertion is a restatement of a previous result. (See §4.) Let R and S be the principal components of $Z-M$, and let F be the frontier of one of them, say R. Since

accessible points are everywhere dense in F and F contains at least one point in every element X, there is an everywhere dense set of points t each of whose corresponding elements contains an accessible point of F.

Orient the points of C, let τ be a fixed point t, and let $\{t_i\}$ and $\{t_i'\}$ be sequences of points $\{t\}$ such that $t_1 < t_2 < \cdots \to \tau$, $t_1' > t_2' > \cdots \to \tau$, and for each t_i and t_i' the corresponding X_i or X_i' contains an accessible point of F.

Let M_i be the arc of M joining X_i and X_i' and containing $X_\tau = f(\tau)$. Obviously $X_\tau = \Pi_1^\infty[M_i]$. By the previous lemma M_i contains a sub-continuum F_i of F joining X_i and X_i', and $(F - F_i) \cdot X_\tau = 0$. Moreover, $\Pi_1^\infty[F_i]$ is a continuum.

But $F_i \subset M_i$; hence $\Pi[F_i] \subset X_\tau$. As $(F - F_i)\ X_\tau = 0$, $F \cdot X_\tau \subset F_i$, whence $F \cdot X_\tau = \Pi[F_i]$. Thus we have shown that for each t, $Y_t = F \cdot X_t$ is a continuum.

On the other hand, $\overline{\lim}_{t \to \tau}$, $Y_t \subset F \cdot X_\tau \subset Y_\tau$. For $Y_t \subset F$ $Y_t \subset X_t$, and $X_t = f(t)$ is upper semi-continuous. Thus $Y_t = g(t)$ is upper semi-continuous. But $f(t)$ is a minimal upper semi-continuous function. Hence for every t, $Y_t = X_t$ and so $F = M$.

COROLLARY. *Let M satisfy the hypotheses of the above theorem, let A and B be any two elements of M, and M_1 and M_2 the complementary arcs of M thus determined. Then $M = H_1 + H_2$, where $H_1 \subset M_1$, $H_2 \subset M_2$, $H_1 \cdot H_2 = \alpha + \beta$, where $\alpha \subset A$, $\beta \subset B$, and both H_1 and H_2 are continua irreducible between α and β.*

PROOF. Let $M = H_1 + H_2$ be an irreducible decomposition* of M such that $H_1 \subset M_1$ and $H_2 \subset M_2$. Then $H_1 \supset M_1 - (A + B)$ and $H_2 \supset M_2 - (A + B)$. By the above theorem M is an irreducible cut of the plane between a point of R and one of S. By a theorem proved elsewhere† H_1 and H_2 are both irreducible between $\alpha = A \cdot H_1 \cdot H_2$ and $\beta = B \cdot H_1 \cdot H_2$.

* The decomposition $M = H_1 + H_2$ is called irreducible if H_1 and H_2 are continua and there exists no proper sub-continuum K of H_1 or H_2 such that $M = K + H_2$ or $M = H_1 + K$, respectively.

† W. A. Wilson, *On irreducible cuts of the plane between two points*, Annals of Mathematics, vol. 29, §9.

7. LEMMA. *Let $M=F(ab)$ be a generalized arc lying in a plane Z and let no element of M separate X_a from X_b. Then $Z-M$ has one component whose frontier has points on every element of M and every other component has a frontier which is a part of some elements.*

PROOF. This is a corollary of a theorem by R. L. Moore (loc. cit., Theorem 9). It is proved in the same manner as the lemma of §4.

8. THEOREM. *Let the aggregate $\{t\}$ be a simple arc ab, let $X=f(t)$ be a minimal upper semi-continuous function defined over ab, and let $M=\Sigma[X]$ lie in the plane Z, while no element of M separates X_a from X_b. Then M is a continuum irreducible between X_a and X_b.*

PROOF. By §7 there is one component R of $Z-M$ which has frontier points on every element of M. Since accessible points are everywhere dense, there is a decreasing sequence $\{t_i\}$ where $t_i \to a$ and an increasing sequence $\{t_i'\}$ where $t_i' \to b$, such that $X_i=f(t_i)$ and $X_i'=f(t_i')$ contain accessible points x_i and x_i', respectively.

Let x_ix_i' be a simple arc lying in R except for the end points. Let u run over a circumference C. Let the segment t_it_i' of ab be homeomorphic with an arc cd of C in such a way that t_i corresponds to c and t_i' to d. Let the arc x_ix_i' be homeomorphic to the complementary arc $dc=\overline{C-cd}$ in such a way that x_i corresponds to c and x_i' to d. Now define the function $Y=g(u)$ as follows. If $u=c$, Y is a sub-continuum of X_i irreducible about x_i and $\overline{\lim}\ X_t$ as $t \to t_i$ in t_it_i'; if $u=d$, Y is a sub-continuum of X_i' irreducible about x_i' and $\overline{\lim}\ X_t$ as $t \to t_i'$ in t_it_i'; if u is any other point of cd and t is the corresponding point of t_it_i', $Y=X_t=f(t)$; if u is a point of $dc-(c+d)$, Y is the corresponding point x of the simple arc x_ix_i'. It is readily seen that $Y=g(u)$ is a minimal upper semi-continuous function and that $N=\Sigma[Y]=G(C)$ is a generalized simple closed curve.

Now Y_c and Y_d determine two generalized arcs $N_1=G(dc)$ and $N_2=G(cd)$ having these elements as ends and no other

common points. If we set H_1 equal to the simple arc x_ix_i' and $H_2=N-x_ix_i'+x_i+x_i'$, it is evident that $H_1 \subset N_1$, $H_2 \subset N_2$, and that $N=H_1+H_2$ is an irreducible partition of N. Then by §6, Corollary, H_2 is irreducible between x_i and x_i'. As $H_2 \supset N_2-(Y_c+Y_d)$, $H_2 \supset X_iX_i'-(X_i+X_i')$. That is, any sub-continuum of the arc X_iX_i' joining the end elements contains all elements between them. Hence any sub-continuum of M joining X_a and X_b contains every point of all the elements between X_i and X_i'. As $t_i \to a$ and $t_i' \to b$, this means that it contains every point of $M-(X_a+X_b)$. But, since $X=f(t)$ is a minimal upper semi-continuous function, $M=\overline{M-(X_a+X_b)}$. Hence M is irreducible between X_a and X_b.

COROLLARY. *Let the aggregate $\{t\}$ be a simple arc ab, let $X=f(t)$ be a minimal upper semi-continuous function defined over ab, and let $M=\Sigma[X]$ lie in the plane Z, while no element of M separates X_a from X_b. Then M is the frontier of one component of $Z-M$ and the frontier of each of the remaining components is a part of some element of M.*

PROOF. By §7 there is one component of $Z-M$ whose frontier has points on every element of M. Since this frontier is a continuum joining X_a and X_b and is a part of M, it must coincide with M by the above theorem. The last part of the corollary is merely restated from §7.

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AN UNNECESSARY CONDITION IN TWO THEOREMS OF ANALYSIS SITUS*

BY P. M. SWINGLE

1. *Introduction.* In this paper K will be a closed point set such that, if S refers to the space under consideration, $S-K=S_1+S_2$. One or more of the following conditions will hold.

CONDITION 1. *Every simple continuous arc which joins a point of S_1 to a point of S_2 contains at least one point of K.*

CONDITION 2. *Every point P of S_i ($i=1$, and 2) can be joined to any point Q of K by a simple continuous arc PQ such that $PQ-Q$ is contained in S_i.*

CONDITION 3. *Every two points of S_i ($i=1$, and 2) can be joined by a simple continuous arc contained in S_i.*

Schoenflies,† using the definition of a simple closed curve given by Jordan, stated and proved the following theorem for a plane S.

THEOREM A. *If K is bounded and if Conditions* 1, 2, *and* 3 *hold, then K is a simple closed curve.*

This theorem was later proved by Lennes‡, using his own definition of a simple closed curve. In his paper *On the foundations of plane analysis situs*, R. L. Moore§ points out that

* Presented to the Society, December 29, 1925.

† A. Schoenflies, *Ueber einen grundlegenden Satz der Analysis Situs*, Göttinger Nachrichten, 1902, p. 185.

‡ N. J. Lennes, *Curves in non-metrical analysis situs with an application in the calculus of variations*, American Journal, vol. 33 (1911), pp. 287–326, Lennes defines a *simple continuous arc* from A to B as a closed, bounded, connected point set containing $A+B$ but containing no proper connected subset that contains $A+B$. He defines a *simple closed curve* as the sum of two simple continuous arcs which have their end points and only their end points in common.

§ Transactions of this Society, vol. 17 (1916), p. 131–164, Theorem 48. This paper will be referred to hereafter as F. A.

this theorem is true in a space S satisfying his set of Axioms Σ_3. J. R. Kline* stated and proved the following theorem for a space S satisfying Σ_3.

THEOREM B. *If K is such that Conditions* 1, 2, *and* 3 *hold, and if S_i ($i=1$, and 2) is unbounded, then K is an open curve.*†

In this paper it is proposed to show that Condition 3 is unnecessary in both Theorem A and Theorem B. The proof will be given on the basis of R. L. Moore's set of Axioms Σ_3. The letter S will be used to represent a space satisfying these axioms. Proofs of these theorems will first be given, using only Conditions 1 and 2, which for Theorem A is independent both of the proof by Schoenflies and of that by Lennes and which for Theorem B is independent of that by Kline. It will then be shown in a very simple manner, independent of these proofs, that Condition 3 is unnecessary in the presence of the other two conditions. I wish to thank Professor R. L. Wilder for his valuable suggestions.

2. *Definitions and Proofs.*

DEFINITION A.‡ *A simple continuous arc from A to B (A and B distinct points) is a closed, bounded,§ connected point set, M, containing $A+B$ such that if X is any point of M distinct from A and B, then $M-X$ is disconnected.*

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* Transactions of this Society, vol. 18 (1917), p. 177–184. Lennes' definition of a simple continuous arc is used.

† An *open curve* is a closed, connected set of points, M, such that if P is a point of M, then $M-P$ is disconnected. A point set is *closed* if it contains all of its own limit points. A set N is *disconnected* (or *not connected*) if there exist point sets U and V such that $N=U+V$ separate. A point set which cannot be so divided is *connected*. By $N=U+V$ *separate* is meant that N is the sum of two non-vacuous mutually exclusive point sets U and V such that neither contains a limit point of the other.

‡ R. L. Moore, *Concerning simple continuous curves*, Transactions of this Society, vol. 21 (1920), pp. 313–320, Def. 2.

§ A point set M is *bounded* if there exists a finite set of regions $R_1, R_2, \cdots, R_n$ such that M is contained in $R_1'+R_2'+\cdots+R_n'$, where $R_i'=R_i+$(the limit points of R_i which are not contained in R_i).

DEFINITION B.* *If AXB and AYB are two simple continuous arcs having only A and B in common, then the point set composed of these two arcs is a simple closed (Jordan) curve.*

DEFINITION C.† *If M is a closed, connected point set which lies in a space S and consists entirely of non-cut points, but which is disconnected by the omission of any two of its points, then M is a simple closed (Jordan) curve.*

DEFINITION. *A region Z will be a region bounded by a simple closed curve C such that $C \times K^{\ddagger} = A_1 + A_2$, where A_1 and A_2 are any two points of K, where $C = A_1X_1A_2 + A_2X_2A_1$, and where $A_1X_iA_2$ $(i = 1, 2)$ is an arc contained in $S_i + (A_1 + A_2)$; and K_1 will represent $K \times Z$ and K_2 will represent $K \times (S - Z')$.*§

THEOREMS 1–47. *These theorems are given in F. A.*

LEMMA I. *If K is such that Condition 1 holds, then there exist at least two distinct points of K.*

Since S is connected, by Theorem 22, and is then a domain, there exists a simple continuous arc, by Theorem 15, joining a point of S_1 to a point of S_2. This arc contains at least one point of K, by Condition 1. Let N be a region containing this point such that $S - N'$ contains a point P of S_1 and a point Q of S_2. Since N' is a closed point set and since $S - N'$ is connected, by Axiom 3, $S - N'$ is a domain and there exists an arc PQ contained in $S - N'$, by Theorem 15. This arc contains at least one point of K, by Condition 1. Therefore there exist at least two distinct points of K.

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*N. J. Lennes, loc. cit.

† This definition is due to J. R. Kline, this Bulletin, vol. 27 (1921), p. 399. R. L. Wilder has proved that this definition is equivalent to Definition B in our space S; see this Bulletin, vol. 31 (1925), p. 224.

‡ If C and K are point sets $C \times K$ is the point set common to C and K.

§ If N is a point set, N' will represent the point set composed of N and its limit points.

LEMMA II. *If K is such that Conditions* 1 *and* 2 *hold, and if A_1 and A_2 are any two points of K, then there exists a region Z, and A_i ($i=1$, and 2) is a limit point of both K_1 and K_2.*

Let W be a point of S_1. There exists an arc A_1W, by Condition 2. such that A_1W-A_1 is contained in S_1 and an arc A_2W such that A_2W-A_2 is contained in S_1. There exists then an arc $A_1Y_1A_2$* such that $A_1Y_1A_2-(A_1+A_2)$ is contained in S_1. Similarly there exists an arc $A_1Y_2A_2$ such that $A_1Y_2A_2-(A_1+A_2)$ is contained in S_2. Let C be the simple closed curve $A_1Y_1A_2+A_2Y_2A_1$. The region bounded by C is then a region Z.

In any region, L, containing A_i ($i=1$, or 2) there exists by Theorem 28 an arc $W_1B_1W_2$ contained in Z (or in $S-Z'$) such that W_j ($j=1$, or 2) is contained in S_j. This arc contains at least one point of K by Condition 1. Therefore A_i is a limit point of K_1 (or of K_2).

LEMMA III. *If there exists a region Z, and if $K_1' = K_{11}+K_{12}$ separate, then there exists a simple closed curve J_{1i} ($i=1$, and 2) enclosing points of K_{1i} and containing at most A_1+A_2 of K and such that $J_{11}\times J_{12}=0$ and J_{1i} is contained in Z'.*

Since $K_1+(A_1+A_2)=K_1'$ is closed and K_{11} and K_{12} are separate, K_{1i} ($i=1$, and 2) is closed. Let P be any point of K_{11}, except A_1 and A_2. P is not a limit point of K_{12} for K_{12} and K_{11} are mutually exclusive closed point sets. There then exists a region, W say, containing P but not containing a point of K_{12}. Then there exists a region, Y say, contained in $Z\times W$, which contains P. Therefore there exists a region, L, by Theorem 5, which contains P and is such that L' is contained in Y. Therefore $L'\times K_{12}=0$. Thus for each point P of $K_{11}-(A_1+A_2)$ there exists a region L containing P such that L' is contained in Z and does not contain a point of K_{12}.

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* Use is here made of the fact: if AB is an arc and K is a closed point set having at least one point in common with AB, then there exists on AB a point belonging to K which precedes every other point of K on AB.

If A_j ($j=1$, or 2) is contained in K_{11} there exists a region, Q_j say, containing the one A_j but not the other and such that $Q_j' \times K_{12}=0$. Then there exists an arc $B_{j1}X_jB_{j2}$ contained in Z', by Theorem 28, such that it has B_{j1} ($j=1$, and 2) and only these points common with C, and such that this arc and the arc $B_{j1}A_jB_{j2}$ of C is contained in Q_j. Let T_{j1} be the region bounded by $B_{j1}X_jB_{j2}+B_{j2}A_jB_{j1}$. It is contained in Z, by Theorem 21. Similarly there exists an arc $C_{j1}Y_jC_{j2}$ contained in $S-Z$, by Theorem 28, but otherwise having the properties of the above arc. Let T_{j2} be the region bounded by $C_{j1}Y_jC_{j2}+C_{j2}A_jC_{j1}$. It is contained in $S-Z'$. Take now for example, the case where $C_{j1}A_jC_{j2}$ contains $B_{j1}A_jB_{j2}$. There exists then, by Theorem 27, a region $T_j=T_{j1}+T_{j2}+(B_{j1}A_jB_{j2}-(B_{j1}+B_{j2}))$. The set T_j' does not contain points of K_{12}.

Since K_{11} is bounded and closed it has the Heine-Borel property,* by Theorem 12. There exists then a finite number of regions, $L_1, L_2, \cdots, L_n$ say, of the set $(L)+(T_i)$ containing all the points of K_{11}. All the above regions can be taken with simple closed curves as boundaries, by Theorem 36. There then exists a subset, G say, of $L_1+L_2+\cdots+L_n$ such that each region of G has a simple closed curve as boundary and such that their sum forms a maximal connected subset of $L_1+L_2+\cdots+L_n$. Replace T_j by T_{j1} in case T_j is a region of G. There exists by Theorem 42 a simple closed curve, J_{11}, which is a subset of the boundaries of the regions of G and which encloses G. The curve J_{11} contains at most A_1+A_2 of K, for only these points of K are on the boundaries of (L) and T_{j1} and not contained in a region of (L).

Since K_{12} and $L_1'+L_2'+\cdots+L_n'$ are mutually exclusive closed point sets, there exists a region, M, for each point, except A_1 and A_2, of K_{12} such that M' is contained in Z and such that it does not contain points of $L_1'+L_2'+\cdots+L_n'$. If A_i ($i=1$, or 2) is contained in K_{12}, take Q_i such that

* A point set is said to have the *Heine-Borel property* if it is true that for every infinite collection H of regions covering it, there exists a finite collection of regions belonging to H and covering it.

Q_i' does not contain points of $L_1'+L_2'+\cdots+L_n'$ and so obtain T_i as above. There then exists a simple closed curve J_{12} enclosing a maximal connected subset, H say, of regions of the set $(M)+(T_{i1})$, where H is composed of a finite number of regions, and J_{12} is a subset of the boundaries of the regions of H.

J_{12} contains at most A_1+A_2 of K and $J_{11}\times J_{12}=0$.

LEMMA IV. *If there exists a region Z, if K_2' is a bounded point set, and if $K_2'=K_{21}+K_{22}$ separate, then there exists a simple closed curve J_{2i} ($i=1$, and 2) enclosing points of K_{2i}, containing at most A_1+A_2 of K, and such that J_{2i} is contained in $S-Z$ and $J_{21}\times J_{22}=0$.*

The proof is the same as in Lemma III.

LEMMA V. *If K is such that Condition 1 holds, and if J is a simple closed curve contaning at most one point, A say, of K, then $J-A$ is contained either in S_1 or in S_2.*

Assume that J contains a point B_i ($i=1$, 2) of S_i. Then J is the sum of two arcs, B_1AB_2 and B_1B_2. Each of these arcs contains at least one point of K, by Condition 1, and therefore J contains at least two distinct points of K, contrary to hypothesis.

LEMMA VI. *If K is such that Condition 2 holds, if J is a simple closed curve contained in S_c ($c=1$, or 2) except for the point set $(A)=J\times K$ which is either vacuous or contains but one point, A, and if J encloses at least one point of K, then J encloses $K-(A)$ and S_i ($i=2$, or 1).*

Let R be the region enclosed by J. Say $S_c=S_1$. Assume that there exists a point, Y say, of S_2 in $S-R'$. Since by hypothesis R contains a point of K, P say, there exists an arc YP, by Condition 2, such that $YP-P$ is contained in S_2. But this is impossible* since J does not contain a point of S_2. Therefore S_2 is contained in R. Assume that there

* Use is here made of the fact: a connected point set which contains a point of a domain, D, and a point not in D, contains a boundary point of D.

exists a point P of K in $S-R'$. Let W be a point of S_2. There exists then an arc PW, by Condition 2, such that $PW-W$ is contained in S_2 which is impossible. Therefore R contains $K-(A)$.

LEMMA VII. *If there exists a region Z, and if K is such that Conditions* 1 *and* 2 *hold, then K_1' is a connected point set.*

Assume that $K_1'=K_{11}+K_{12}$ separate. Then, by Lemma III, there exists a simple closed curve J_{1i} ($i=1$, and 2) enclosing points of K_{1i} and containing at most A_1+A_2 of K and such that $J_{11}\times J_{12}=0$. J_{11} may contain A_1, in which case J_{12} will contain at most A_2 of K. Therefore in every possible case one of these curves contains at most one point of K. Say J_{11} does. Then J_{11}, by Lemma V, is contained, except for possibly one point, in S_i ($i=1$, or 2) and so, by Lemma VI, encloses all except possibly one point of K. But it is shown in Lemma II that K_2 is non-vacuous and hence there are points of K in $S-Z'$. But J_{11} is contained in Z' and so K is. Therefore our assumption is incorrect and so K_1' is a connected point set.

LEMMA VIII. *If there exists a region Z, and if K is such that Conditions* 1 *and* 2 *hold, and if K_2' is bounded, then K_2' is a connected point set.*

Assume that $K_2'=K_{21}+K_{22}$ separate. Then there exists a simple closed curve J_{2i} ($i=1$, and 2) enclosing points of K_{2i}, contained in $S-Z$, containing at most A_1+A_2 of K, and such that $J_{21}\times J_{22}=0$, by Lemma IV. If one curve contains A_1+A_2 the other one does not contain a point of K; if one contains at most one point of K the other contains at most one point of K. Consider, for example, the case where J_{21} contains A_1+A_2, and let R_{21} be the region bounded by this curve. Consider now how J_{21} was obtained in Lemma IV. If it contains A_i it must contain the arc $C_{i1}A_iC_{i2}$ which is common to the boundaries of the region T_{i2} and the region Z, for only the boundaries of these two regions contained A_i in Lemma IV. Thus there exists a simple closed curve,

J_{21}'' say, which is a subset of C and J_{21} and which encloses Z and R_{21}, and also encloses T_1 and T_2. Therefore J_{21}'' does not contain A_i ($i=1$, or 2). Thus in each possible case we obtain two curves each contained in S_j ($j=1$, or 2) except for possibly one point, by Lemma V. Then each curve encloses all except possibly one point of K, by Lemma VI. Consider, for example, the case where J_{21} and J_{22} have this property. Since they have no points in common and since each encloses all except possibly one point of K, one must enclose the other. Say J_{21} encloses J_{22}. Consider again how J_{21} was obtained in Lemma IV. It is composed of the boundaries of regions each of which, among other properties, contains points of K_{21}. Say N is one of these regions and that it contains the point P of K_{21}. N' does not contain a point of J_{22} for N was so selected in the proof of Lemma IV. Let R_{22} be the region bounded by J_{22}. Since N has boundary points in $S-R_{22}'$ and since it does not contain boundary points of R_{22}, it, and so points of K, are contained in $S-R_{22}'$. Thus J_{22} does not enclose all except possibly one point of K on J_{22}. Therefore K_2' is a connected point set.

LEMMA IX. *If there exists a region Z, if K_2' is bounded, and if K is such that Conditions* 1 *and* 2 *hold, then the omission of any point P_i ($i=1$, and 2) of K_i', $P_i \neq A_1$ or A_2, disconnects K_i'.*

There exists an arc $Q_{i1}P_iQ_{i2}$ which is, except for its end points, contained in Z or in $S-Z'$ according as $i=1$ or 2, where $Q_{ij}P_i-P_j$ ($j=1$, or 2) is contained in S_j, by Condition 2. The arc $Q_{11}P_1Q_{12}$, by Theorem 25, divides Z into two regions, Z_{11} and Z_{12} say, where Z_{ij}' ($j=1$, and 2) contains A_j. Let $KX(Z_{1n}+A_n)=K_{1n}$. The sets K_{1n} are non-vacuous by Lemma II. Therefore $K_1'-P_1=K_{11}+K_{12}$ separate. Also $Q_{21}Q_{22}$ of C and $Q_{21}P^2Q_{22}$ enclose a region, Z'' say. Take, for example, the case where $Q_{21}Q_{22}$ contains A_1. Let $Z''\times K_2+A_1=K_{21}$ and $(S-Z'')\times K_2+A_2=K_{22}$. Then $K_2'-P_2=K_{21}+K_{22}$ separate.

LEMMA X. *If there exists a region Z, and if K is bounded and such that Conditions* 1 *and* 2 *hold, then K_i' ($i=1$, or 2) is a simple continuous arc.*

K_i' is bounded and closed by hypothesis, connected by Lemma VII and Lemma VIII, and disconnected by the omission of any point other than A_1 or A_2, by Lemma IX. Therefore, by Definition A, K_i' is a simple continuous arc with end points A_1 and A_2.

THEOREM I. *This is Theorem* A *with Condition* 3 *omitted.*

By Lemma I, there exist two points of K. Then, by Lemma II, there exists a region Z. Thus, by Lemma X, K is the sum of two simple continuous arcs having only their end points in common. Therefore, by Definition B, K is a simple closed curve.

LEMMA XI. *If K is such that Condition* 1 *holds, and if S_i ($i=1$, and* 2*) is unbounded, then K is not a simple closed curve.*

Assume that K is a simple closed curve, enclosing the region R. There exists a point of S_1 and a point of S_2 in $S-R'$, for S_i ($i=1$, and 2) is unbounded. Then there exists an arc joining these two points and contained in $S-R'$. This arc does not contain a point of K, contrary to hypothesis. Therefore K is not a simple closed curve.

THEOREM II. *This is the same as Theorem* B *with Condition* 3 *omitted.*

There exist at least two distinct points of K by Lemma I, P_1 and P_2 say. Assume that $K=N_1+N_2$ separate and let A_i ($i=1$, and 2) be a point of N_i. There exists a region Z by Lemma II. Then, by Lemma VII, there exists a connected subset of K, contained in Z', and containing A_1+A_2. But that this connected subset of K should contain points of both of the separate point sets N_1 and N_2 is impossible. Therefore our assumption is incorrect and so K is a connected point set.

K is disconnected by the omission of any two points, A_1 and A_2. For there exists a region Z, by Lemma II, having these two points and only these two points of K on its boundary and such that Z and $S-Z'$ each contain points of K. Thus the omission of these two points divides K into two separate point sets $K\times Z$ and $K\times(S-Z')$. K is a closed point set by hypothesis. Thus by Definition C, if K consists entirely of non-cut points, K is a simple closed curve, contrary to Lemma XI. Thus there exists a cut point, P say, of K. Let $K-P=M_1+M_2$.

Let A_i ($i=1$, and 2) be any point of M_i. Then there exists a region Z by Lemma II, which, by Lemma VII, contains a connected subset of K joining A_1 and A_2. That is, this connected point set joins points of the two separate point sets M_1 and M_2. This is impossible unless it contains P. Just as in Lemma X, K_1' being bounded here is an arc. Both M_1 and M_2 have points in Z as P is clearly a limit point of both. Now $K_1'-P$ is the sum of two connected sets, each of which can lie, obviously, in only one of the sets M_1, M_2. Thus it is seen that one of these connected sets must be contained in M_1 and one in M_2. Let $M_i\times(Z+A_i)=M_{i1}$ and $M_i\times(S-Z')=M_{i2}$. These point sets are non-vacuous by Lemma II.

The sets M_{12} and M_2 are separate since M_1 and M_2 are separate. Also $(M_{11}-A_1)$ and M_{12} are separate for Z and $S-Z'$ are separate. P is not a limit point of $S-Z$ and so is not a limit point of M_{12}. Therefore $K-A_1=M_{12}+(P+(M_{11}-A_1)+M_2)$ separate. Therefore the omission of any point of M_1 disconnects K. Similarly for any point of M_2. Therefore the omission of any point of K disconnects K.

Let P be any point of K and let $K-P=N_1+N_2$ separate. Assume that $N_1=N_{11}+N_{12}$ separate. Let A_i ($i=1$, and 2) be any point of N_{1i}. There exists by Lemma II a region Z such that $K\times Z'=A_1A_2$ is a simple continuous arc as in Lemma X. This arc joins points of the two separate point sets N_{11} and N_{12}. This is impossible unless it contains P. Thus this arc is the sum of two arcs PA_1 and PA_2. The arc

PA_1 cannot contain points of N_2 for it would then contain a connected point set containing points of the two separate point sets N_1 and N_2. Similarly, it cannot contain points of N_{12}. Therefore PA_1-P must be contained in N_{11}. Similarly, PA_2-P must be contained in N_{12}. Therefore the arc A_1A_2-P must be contained in N_1. Therefore the region Z does not contain points of N_2. Thus P is not a limit point of N_2. But it is well known that N_2+P is a connected point set and so P is a limit point of N_2. Therefore our assumption is incorrect and so N_i ($i=1$, and 2) is a connected point set.

K is closed by hypothesis. Therefore K is an open curve.

A simple proof that Condition 3 is unnecessary in Theorems A and B may also be obtained from Lemma I, Lemma II, and the following theorem.

THEOREM III. *If L is a subset of a point set M, and if $M'-L=M_1+M_2$ such that* (2) *every point of M_i ($i=1$ and 2) can be joined to any point Q of L by a simple continuous arc PQ such that $PQ-Q$ is contained in M_i, and* (3) *M_j ($j=1$, or 2) is not arc-wise connected, L contains at most two points.*

Take for example the case where M_1 is not arc-wise connected, there being no arc AB of M_1 joining the two points A and B of M_1. Assume that L contains at least three points, E_i ($i=1$, 2, and 3) say. Then by Condition 2 there exist three arcs AE_i and three arcs BE_i contained in $M_1+(E_1+E_2+E_3)$. Then there exist three arcs, A_1E_i, distinct except for A_1, contained in $AE_1+AE_2+AE_3$ and three arcs B_1E_i, distinct except for B_1, contained in $BE_1+BE_2+BE_3$. These two sets of three arcs have but $E_1+E_2+E_3$ in common for otherwise there exists an arc AB in M_1. There exist then three arcs $A_1E_iB_1$ ($i=1$, 2, 3) having only A_1 and B_1 in common. Thus there exist three simple closed curves $A_1E_1B_1+A_1E_2B_1=J_1$ say, $A_1E_2B_1+A_1E_3B_1=J_2$ say, and $A_1E_3B_1+A_1E_1B_1=J_3$ say. Let R_i ($i=1$, 2, and 3) be the region bounded by J_i. Consider for example the case where $R_1=R_2+R_3+A_1E_3B_1-(A_1+B_1)$. J_1 is contained in M_1+L.

Assume that $S-R_1'$ contains a point, say U, of M_2. Then by Condition 2 there exists an arc UE_3 contained in M_2+E_3. This is impossible for such an arc must contain a point of J_1. Therefore M_2 is contained in R_1. Similarly it can be proved that M_2 is contained in $S-R_2'$ and so in R_3. Similarly it can be proved that M_2 is contained in $S-R_3'$. Thus a contradiction is obtained. Therefore there do not exist three points of L.

CORNELL UNIVERSITY

TWO CLASSES OF PERIODIC ORBITS WITH REPELLING FORCES*

BY T. H. RAWLES

1. *Introduction.* The purpose of this paper is to obtain certain classes of periodic orbits† of a system consisting of one body of very great mass which attracts two mutually repellent bodies of very small mass. The forces of attraction and repulsion in the system are assumed to vary inversely as the squares of the distances. It will be shown that if certain conditions of symmetry are imposed the orbits of the two small bodies must lie either in two parallel planes or else be coplanar. In the former case they are circles about a line perpendicular to the two planes and passing through the large body. For the second type of orbit a development in powers of a parameter will be obtained.

2. *The Equation of Motion.* By a proper choice of the units of time and space we can make the coefficient of the intensity of the attraction equal to 1. Then if K^2 is the

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† The orbits referred to were first calculated by Langmuir, who employed numerical integration. (Physical Review, vol. 17 (1921), p. 339.) They have been discussed by Van Vleck in his monograph on *Quantum Principles and Line Spectra*. (Bulletin of the National Research Council, Number 54, p. 89.)

ratio of the repulsion to the attraction the force function of the system is

$$U = \frac{1}{r_1} + \frac{1}{r_2} - \frac{K^2}{\Delta}, \tag{1}$$

where r_1 and r_2 represent the distances between the two small bodies and the large body while Δ is the distance between the small bodies.

The unit of mass may be so chosen that the mass of each of the small bodies is unity. Then let m denote the mass of the large body. The system will be referred to the rectangular axes ξ, η, ζ, and we shall indicate the coordinates of the large body by the subscript 0, and the coordinates of the small bodies by the subscripts 1 and 2. With this choice of units and coordinates, the equations of motion are

$$m\ddot{\xi}_0 = \frac{\partial U}{\partial \xi_0}, \quad m\ddot{\eta}_0 = \frac{\partial U}{\partial \eta_0}, \quad m\ddot{\zeta}_0 = \frac{\partial U}{\partial \zeta_0}; \tag{2}$$

$$\ddot{\xi}_i = \frac{\partial U}{\partial \xi_i}, \quad \ddot{\eta}_i = \frac{\partial U}{\partial \eta_i}, \quad \ddot{\zeta}_i = \frac{\partial U}{\partial \zeta_i}, \quad (i = 1,2). \tag{3}$$

If m is very large, the effect of the attraction of the other bodies on the large body is very small. We shall assume that this can be neglected altogether. In other words we assume that the mass of the large body is infinite as compared to that of the other two. Under this assumption we can neglect the right members of (2). These equations then show that the large body moves through space with uniform velocity in a straight line.

We now take this body as the origin, and denote the coordinates of the other two by x_i, y_i, z_i, $(i=1, 2.)$ The equations of motion of the two small bodies then become

$$\ddot{x}_i = \frac{\partial U}{\partial x_i}, \quad \ddot{y}_i = \frac{\partial U}{\partial y_i}, \quad \ddot{z}_i = \frac{\partial U}{\partial z_i}, \tag{4}$$

where

$$U = \frac{1}{r_1} + \frac{1}{r_2} - \frac{K^2}{\Delta}, \tag{5}$$

and

$$(6)\qquad \begin{cases} r_i^2 = x_i^2 + y_i^2 + z_i^2, & (i = 1,2), \\ \Delta^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2. \end{cases}$$

3. *Periodic Orbits.* The periodic orbits with which we are concerned are those for which

$$(7)\qquad y_1 - y_2 = 0, \quad z_1 - z_2 = 0, \quad x_1 - x_2 = 0.$$

With these restrictions equations (4) now become

$$(8)\qquad \begin{cases} \ddot{x}_1 = \dfrac{-x_1}{r_1^3} + \dfrac{k^2(x_1 - x_2)}{((x_1 - x_2)^2)^{3/2}} = \dfrac{-x_1}{r_1^3} + \dfrac{k^2}{(2x_1)^2}, \\ \ddot{y}_1 = \dfrac{-y_1}{r_1^3}, \quad \ddot{z}_1 = \dfrac{-z_1}{r_1^3}, \end{cases}$$

with three similar equations for the coordinates of the other particle. The variables are now separated and we need only consider equations (8). When these are solved the motion of the other particle can be obtained from (7).

Let us now put $x_1 = x$, $y_1 = y$, $z_1 = z$, $r_1 = r$, and $k^2/4 = 1 - e^2$. Our equations may then be written in the form

$$(9)\qquad \ddot{x} = \frac{-x}{r^3} + \frac{1 - e^2}{x^2}, \quad \ddot{y} = \frac{-y}{r^3}, \quad \ddot{z} = \frac{-z}{r^3}.$$

Two integrals of (9) can be found at once. They are

$$(10)\qquad \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{r} - \frac{1 - e^2}{x} + C, \quad \dot{z}y - z\dot{y} = h.$$

Also we notice that if a solution

$$x = X(e,t), \quad y = Y(e,t), \quad z = Z(e,t)$$

is known, then another solution is

$$x = aX\left(e, \frac{t - t_0}{a^{3/2}}\right), \quad y = aY\left(e, \frac{t - t_0}{a^{3/2}}\right), \quad z = aZ\left(e, \frac{t - t_0}{a^{3/2}}\right).$$

4. *Periodic Solutions of Period* 2π. We shall obtain solutions periodic with the period 2π. From the remark which we have just made we can infer the existence of solutions of any desired period. The existence of a solution developed in powers of e as a parameter follows from a theorem of MacMillan.* When $e=0$, a solution of the equations is

$$x = 1, \quad y = 0, \quad z = 0. \tag{11}$$

The solutions which we shall proceed to develop are of the form

$$\begin{cases} x = 1 + e^2x_2 + e^4x_4 + \cdots, \\ y = ey_1 + e^3y_3 + \cdots, \\ z = ez_1 + e^3z_3 + \cdots. \end{cases}$$

When developed in powers of the parameter, the equations of motion are

$$\begin{cases} \sum_0 e^{2i}\ddot{x}_{2i} = [\tfrac{3}{2}(y_1^2 + z_1^2) - 1]e^2 + \cdots, \\ \sum_0 e^{2i+1}\ddot{y}_{2i+1} = -y_1e + [-y_3 \\ \qquad\qquad + \tfrac{3}{2}y_1(2x_2 + y_1^2 + z_1^2)]e^3 + \cdots, \\ \sum_0 e^{2i+1}\ddot{z}_{2i+1} = -z_1e + [-z_3 \\ \qquad\qquad + \tfrac{3}{2}z_1(2x_2 + y_1^2 + z_1^2)]e^3 + \cdots. \end{cases} \tag{12}$$

Equating the coefficients of e in the last two equations we obtain

$$\ddot{y}_1 + y_1 = 0, \qquad \ddot{z}_1 + z_1 = 0. \tag{13}$$

The solutions of these equations are

$$\begin{cases} y_1 = c_1 \cos t + c_2 \sin t, \\ z_1 = c_3 \cos t + c_4 \sin t, \end{cases} \tag{14}$$

where $c_1, \cdots, c_4$ are constants of integration. From the coefficients of e in the first of (12), we have

* Transactions of this Society, vol. 13 (1912), p. 157.

$$(15)\quad \begin{cases} \ddot{x}_2 = \dfrac{3}{2}(y_1^2 + z_1^2) - 1 \\ \quad = \dfrac{3}{2}\left[\left(\dfrac{c_1^2 + c_2^2 + c_3^2 + c_4^2}{2}\right)\right. \\ \qquad + \left(\dfrac{c_1^2 + c_3^2 - c_2^2 - c_4^2}{2}\right)\cos 2t \\ \qquad \left. + (c_1c_2 + c_3c_4)\sin 2t\right] - 1. \end{cases}$$

Upon integration we obtain

$$x_2 = \left(\frac{3}{4}(c_1^2 + c_2^2 + c_3^2 + c_4^2) - 1\right)\frac{t^2}{2} + c_5 t + c_6$$
$$- \frac{3}{16}(c_1^2 + c_3^2 - c_2^2 - c_4^2)\cos 2t - \frac{3}{8}(c_1c_2 + c_3c_4)\sin 2t,$$

c_5 and c_6 being arbitrary.

The periodicity conditions which we shall impose are that $x(2\pi) - x(0) = 0$ and $\dot{x}(2\pi) - \dot{x}(0) = 0$, with similar conditions for the other coordinates. Moreover, the coefficient of each power of the parameter must satisfy these conditions. When the periodicity conditions are applied to x_2 and $\dot{x}_2$ we see that

$$\left(\frac{3}{4}(c_1^2 + c_2^2 + c_3^2 + c_4^2) - 1\right)\frac{(2\pi)^2}{2} - 2\pi c_5 = 0,$$

$$\left(\frac{3}{4}(c_1^2 + c_2^2 + c_3^2 + c_4^2) - 1\right)(2\pi) = 0.$$

From these relations we now find that

$$(16)\qquad c_1^2 + c_2^2 + c_3^2 + c_4^2 = 4/3, \quad c_5 = 0.$$

Obviously the periodicity conditions imposed on y_1, $\dot{y}_1$, z_1 and $\dot{z}_1$ are satisfied identically.

When we proceed to the next approximation we have the equations

$$(17)\qquad \begin{cases} \ddot{y}_3 + y_3 = (3/2)(2x_2 + y_1^2 + z_1^2)y_1, \\ \ddot{z}_3 + z_3 = (3/2)(2x_2 + y_1^2 + z_1^2)z_1. \end{cases}$$

These equations become

$$\ddot{y}_3 + y_3 = \sum_i A_i^{(3)} \sin it + \sum B_i^{(3)} \cos it,$$

$$\ddot{z}_3 + z_3 = \sum_i C_i^{(3)} \sin it + \sum D_i^{(3)} \cos it, \quad (i = 1,3),$$

where the coefficients of the periodic terms represent known functions of the constants of integration. Upon integrating the first of these two equations, we have

$$y_3 = c_1^{(3)} \cos t + c_2^{(3)} \sin t - A_1^{(3)} \frac{t \cos t}{2} + B_1^{(3)} \frac{t \sin t}{2} + \text{terms in } \cos 3t \text{ and } \sin 3t,$$

where $c_1^{(3)}$ and $c_2^{(3)}$ are the new constants of integration. When we impose the condition that y_3 and $\dot{y}_3$ shall be periodic we find that $A_1^{(3)} = B_1^{(3)} = 0$. In the same manner we can show that $C_1^{(3)} = D_1^{(3)} = 0$.

5. *Restrictions on the Constants.* These four equations impose new restrictions on the constants c_i. They become

$$(18)\qquad \begin{cases} B_1^{(3)} = 3(c_6 + \tfrac{1}{3})c_1 + c_1 M + c_2 N = 0, \\ A_1^{(3)} = 3(c_6 + \tfrac{1}{3})c_2 - c_2 M + c_1 N = 0, \\ D_1^{(3)} = 3(c_6 + \tfrac{1}{3})c_3 + c_3 M + c_4 N = 0, \\ C_1^{(3)} = 3(c_6 + \tfrac{1}{3})c_4 - c_4 M + c_3 N = 0, \end{cases}$$

where

$$M = \tfrac{3}{32}(c_1^2 + c_3^2 - c_2^2 - c_4^2), \quad N = \tfrac{3}{16}(c_1 c_2 + c_3 c_4).$$

When we multiply the four equations by c_4, $-c_3$, $-c_2$, c_1, respectively and add, the result is

$$(19)\qquad 3(c_6 + \tfrac{1}{3})(c_1 c_4 - c_2 c_3) = 0.$$

There are three cases according as one factor is zero or both factors are zero.

CASE I. $c_6+1/3=0$, $c_1c_4-c_2c_3\neq 0$.
The first of (16) shows that not all of the constants can be zero. It then follows from equations (18) that $M=N=0$. This gives us the two equations

$$c_1^2 + c_3^2 - c_2^2 - c_4^2 = 0, \quad c_1c_2 + c_3c_4 = 0.$$

If, for example, $c_2\neq 0$ we have from the second of these two equations

$$c_1^2 = \frac{c_3^2c_4^2}{c_2^2}.$$

When this is substituted in the first it becomes

$$c_3^2c_4^2 + c_2^2c_3^2 - c_2^4 - c_4^2c_2^2 = 0,$$

or

$$(c_2^2 + c_4^2)(c_3^2 - c_2^2) = 0.$$

Since $c_2\neq 0$ we must have

$$c_3 = c_2, \quad c_1 = -c_4, \quad \text{or} \quad c_1 = c_4, \quad c_3 = -c_2.$$

Case I shows that as far as the approximation has been carried the motion is in a circle in a plane normal to the x axis. In fact we have

$$y_1^2 + z_1^2 = c_1^2 + c_2^2 = \tfrac{2}{3},$$
$$x_2 = -\tfrac{1}{3}.$$

CASE II. $c_6+1/3=0$, $c_2c_3-c_1c_4=0$.
It is easy to show that this case is impossible. We would have as in Case I $M=N=0$, and would obtain the same relations between the constants. When these relations are substituted in $c_2c_3-c_1c_4=0$ that equation becomes $c_2^2+c_1^2=0$, which is impossible since not all of these constants can be zero.

CASE III. $c_6+1/3\neq 0$, $c_2c_3-c_1c_4=0$.
On multiplying the first of equations (14) by c_3, the second by c_1 and subtracting, since $c_2c_3-c_1c_4=0$, we obtain $c_3y_1 - c_1z_1=0$, which shows that the motion lies in a plane passing through the x axis. (See Fig. 1.)

6. *The Orbits.* We shall next show that when the process is continued each approximation gives rise to terms of the same type as those already found, and, therefore, that the periodic orbits are either circles about the x axis, or else they lie entirely in a plane containing the x axis.

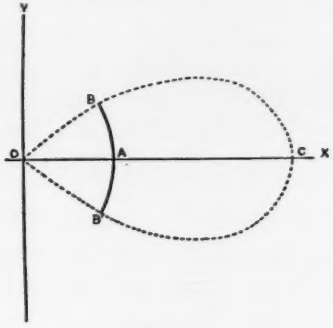

FIG. 1

The figure illustrates the path described by one of the particles for the case in which $e=1/2$. When $t=0$ it is at A. It reaches B when $t=\pi/2$, stops, and then returns along the same path to A, $t=\pi$. It then continues to B', $t=3\pi/2$, and finally completes the cycle by returning to A when $t=2\pi$. The position of the other particle is at all times symmetric to that of this one with respect to the y axis. The curve of zero velocity (conchoid) is indicated by $OBCB'O$.

For the discussion of Case I, let us suppose that when $t=0$, $y=0$. We then have

$$c_2 = c_3 = (\tfrac{2}{3})^{1/2}, \quad c_4 = -c_1 = 0.$$

The solution, as far as we have carried it, is

$$y = e(c_2 \sin t) + e^3(c_2^{(3)} \sin t),$$
$$z = e(c_2 \cos t) + e^3(c_3^{(3)} \cos t + c_4^{(3)} \sin t),$$
$$x = 1 + e^2(-\tfrac{1}{3}).$$

The equation for x_4 becomes

$$\ddot{x}_4 = 3(y_1y_3 + z_1z_3) + 2x_2 + \tfrac{3}{2}x_2^2$$
$$= (\tfrac{3}{2}(c_2c_2^{(3)} + c_2c_3^{(3)}) - \tfrac{1}{6}) + \tfrac{3}{2}\{c_2(c_3^{(3)} - c_2^{(3)})\}\cos 2t + \tfrac{3}{2}c_2c_4^{(3)} \sin 2t.$$

Periodicity requires that

$$c_2(c_2^{(3)} + c_3^{(3)}) = 1/9,$$

and we have upon integration

$$x_4 = c_6^{(4)} - \tfrac{3}{8}c_2(c_3^{(3)} - c_2^{(3)}) \cos 2t - \tfrac{3}{8}8c_2c_4^{(3)} \sin 2t.$$

The equations for y_5 and z_5 now become

$$\ddot{y}_5 + y_5 = \tfrac{3}{2}y_1(2x_4 + x_2^2 + 2(y_1y_3 + z_1z_3)),$$
$$\ddot{z}_5 + z_5 = \tfrac{3}{2}z_1(2x_4 + x_2^2 + 2(y_1y_3 + z_1z_3)),$$

which are when written out in full

$$\ddot{y}_5 + y_5 = \tfrac{3}{2}(c_2 \sin t)\{(\tfrac{3}{5} + 2c_6^{(4)}) + \tfrac{1}{4}c_2(c_3^{(3)} - c_2^{(3)}) \cos 2t + \tfrac{1}{4}c_2c_4^{(3)} \sin 2t\},$$
$$\ddot{z}_5 + z_5 = \tfrac{3}{2}(c_2 \cos t)\{(\tfrac{3}{5} + 2c_6^{(4)}) + \tfrac{1}{4}c_2(c_3^{(3)} - c_2^{(3)}) \cos 2t + \tfrac{1}{4}c_2c_4^{(3)} \sin 2t\}.$$

When the products of the trigonometric functions are expressed as functions of multiple angles it will be seen that for periodicity we must have

$$c_6^{(4)} = -1/9, \quad c_3^{(3)} = c_2^{(3)}, \quad c_4^{(3)} = 0.$$

The last of these conditions shows that z_3 contains no term in sin t. The complementary function for z_5 may contain a term in sin t but the next integration will show that the coefficient of this term is zero.

The process can be continued indefinitely and our solution is of the form

$$y = p_1 \sin t, \quad z = p_1 \cos t, \quad x = p_2,$$

where p_1 and p_2 are power series in e. The orbit is obviously a circle about the x axis.

We could have obtained this orbit by more elementary means. If we put

$$x = (1 - e^2)^{1/3},$$
$$y = (1 - (1 - e^2)^{2/3})^{1/2} \sin (t - t_0),$$
$$z = (1 - (1 - e^2)^{2/3})^{1/2} \cos (t - t_0),$$

it is easily seen that the equations of motion are satisfied.

For the discussion of the case where the motion, as far as the first approximation is concerned, lies in a plane containing the x axis, let us choose that plane $z=0$. We shall take as the origin of time the instant when $y=0$. With this choice we have $c_2=(4/3)^{1/2}$, $c_1=c_3=c_4=0$.

Under these restrictions the equation for z_3 is $\ddot{z}_3+z_3=0$, and its solution is

$$z_3 = c_3^{(3)} \cos t + c_4^{(3)} \sin t.$$

The constants of integration introduced in the complementary function of x_2 have already been determined by (16) and (18), and the equation for z_5 now becomes

$$\ddot{z}_5 + z_5 = z_3(3x_2 + \tfrac{3}{2}y_1^2)$$
$$= -\tfrac{1}{4}c_3^{(3)} \cos t - \tfrac{1}{8}c_3^{(3)} \cos 3t - \tfrac{1}{8}c_4^{(3)} \sin 3t.$$

Now for a periodic solution it is obvious that $c_3^{(3)}=0$. Therefore z_3 contains only a term in sin t. If we continue this process imposing each time the condition that when $t=0$, $y=0$, we shall find that z as well as y contains only terms in sin $(2k+1)t$. This means of course that the particle crosses the x axis when $t=n\pi$. The motion will always remain in the plane which contains the x axis and the tangent to the orbit at the instant the particle crosses that axis because all the forces are acting in this plane. This can be easily verified by a consideration of the differential equations.

7. *Indefinite Extension of the Process.* It remains to show that the process of determining the constants can be carried on indefinitely. For this purpose we may take $z=0$ as the

plane of motion and suppose that the following terms have been calculated:

$$(20)\qquad \begin{cases} x_{2j} = \sum_{0}^{j} a_{2i}^{(2j)} \cos 2it, & (j = 0,1,2, \cdots, n), \\ y_{2j+1} = d_{2j+1} \sin t + \sum_{1}^{j} a_{2i+1}^{(2j+1)} \sin (2i+1)t. \end{cases}$$

All the coefficients have been determined except d_{2n+1}. When we equate the coefficients of e^{2n+2} in the equation for x we find

$$\ddot{x}_{2n+2} = x_{2n+2}(-1+3-2) + 3y_1 y_{2n+1} + \text{known terms}.$$

This may be written in the form

$$(21)\qquad \ddot{x}_{2n+2} = \tfrac{3}{2}c_2 d_{2n+1}(1 - \cos 2t) + \sum_{0}^{n+1} A_{2i}^{(2n+2)} \cos 2it,$$

where all the coefficients are known except d_{2n+1}. The arguments of the periodic functions must be even multiples of the time for they all enter from x or y^2 which contain only terms with arguments of that form.

The integration of this equation will produce a secular term unless

$$(\text{i})\qquad \tfrac{3}{2}c_2 d_{2n+1} + A_0^{(2n+2)} = 0.$$

Having imposed this condition, we find that

$$(22)\qquad x_{2n+2} = b_{2n+2}t + d_{2n+2} + \sum_{1}^{n+1} a_{2i}^{(2n+2)} \cos 2it,$$

where b_{2n+2} and d_{2n+2} are constants of integration and

$$a_{2i}^{(2n+2)} = \frac{-A_{2i}^{(2n+2)}}{(2i)^2}.$$

For periodicity we must of course put

$$(\text{ii})\qquad b_{2n+2} = 0.$$

We now determine y_{2n+3} by equating the coefficients of e^{2n+3} in the y equation. This gives

$$\ddot{y}_{2n+3} + y_{2n+3} = 3y_1x_{2n+2} + \text{known terms},$$

which may be written in the form

$$\ddot{y}_{2n+3} + y_{2n+3} = 3c_2d_{2n+2} \sin t + \sum_{0}^{n+1} A_{2i+1}^{(2n+3)} \sin (2i+1)t. \tag{23}$$

The arguments of the periodic functions are odd multiples of the time because they arise from terms such as y^{2k+1} and $y^{2k+1}x^{2h}$. If the coefficient of $\sin t$ in the right member is not zero a secular term will be introduced. This requires that

$$3c_2d_{2n+2} + A_1^{2n+3} = 0. \tag{iii}$$

When the equation is integrated we have

$$y_{2n+3} = d_{2n+3} \sin t + b_{2n+3} \cos t + \sum_{1}^{n+1} a_{2i+1}^{(2n+3)} \sin (2i+1)t, \tag{24}$$

where

$$a_{2i+1}^{(2n+3)} = \frac{A_{2i+1}^{(2n+3)}}{1-(2i+1)^2}.$$

We may suppose that the origin of time has been fixed by the condition $y=0$ when $t=0$. This determines the remaining constant and we have

$$b_{2n+3} = 0. \tag{iv}$$

We now have determined all the coefficients to and including x_{2n+2} and y_{2n+2}, except the coefficient of $\sin t$ in the latter. Equations (22) and (24) are of the same form as (20), with n replaced by $n+1$ and it is evident that the process can be carried on indefinitely.

8. *Development of x and y in Terms of the Parameter.* The development of x and y has been carried as far as terms involving the fifth power of the parameter. The expressions found are

$$
(25)\qquad \begin{cases} x = 1 + \left(-\dfrac{3}{8} + \dfrac{1}{4}\cos 2t \right) e^2 \\ \qquad + \left(-\dfrac{391}{4608} + \dfrac{29}{384}\cos 2t - \dfrac{5}{1536}\cos 4t \right) e^4, \\ y = (3)^{1/2}\left[\left(\dfrac{2}{3}\sin t \right) e + \left(\dfrac{1}{96}\sin 3t \right) e^3 \right. \\ \qquad + \left(-\dfrac{3307}{331776}\sin t + \dfrac{91}{36864}\sin 3t \right. \\ \qquad \left. \left. + \dfrac{37}{110592}\sin 5t \right) e^5 \right]. \end{cases}
$$

For the case of the helium atom $k^2 = 1/2$ and the parameter e must be given the value $(7/8)^{1/2}$. For this value of the parameter the convergence is not rapid, and the terms which we have calculated do not give the same degree of accuracy as the numerical integration of Langmuir. However the two results are in sufficient agreement to verify each other.

The numerical results are easily checked by means of the energy integral

$$
(10)\qquad \frac{1}{2}(\dot{x}^2 + \dot{y}^2) = \frac{1}{r} - \frac{1 - e^2}{x} + C.
$$

The constant of integration is itself a power series in the parameter. As far as the sixth power of the parameter, it is

$$
(26)\qquad C = -\frac{1}{3}e^2 + \frac{1}{16}e^4 + \frac{127}{10368}e^6.
$$

YALE UNIVERSITY

RATIONAL QUINTIC SURFACES WITH TWO SKEW DOUBLE LINES*

BY A. R. WILLIAMS

The purpose of this paper is to obtain by different methods, and where possible to extend, the results given by Clebsch and Hill† for quintic surfaces having two distinct skew double lines. In a second paper, which will appear in an early issue of this Bulletin, I discuss a surface, due to Montesano, which has two consecutive skew double lines.

The equation of a quintic surface that has xy and zw for double lines, but not further specialized, contains twenty-four terms. If we take a point on each of them, say $(0:0:z_1:1)$ and $(x_1:1:0:0)$ we find that for a given surface there are thirteen pairs of values of x_1 and z_1 such that the line joining corresponding points lies in the surface. If we allow the coefficients to vary we find that eleven of these transversals may be assigned arbitrarily and the remaining two are then determined. That is, there is a single infinity of such surfaces all of which have the eleven transversals and the other two determined by them. The form of the equation shows that there are six pinch points on each of the double lines.

The surface is obviously rational, that is, its points may be put in one to one correspondence with the points of a plane. For the line drawn from any point P of the surface to intersect the two double lines meets a given plane, π_0, in a point P', which we may consider the image of P. A section of the quintic surface has of course two double points where its plane cuts the double lines. Therefore, establishing the correspondence in the way just indicated, as P moves on the plane section the surface generated by the

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* Presented to the Society, San Francisco Section, April 2, 1927.

† Clebsch, Mathematische Annalen, vol. 1, p. 253.
Hill, Mathematical Review, vol. 1, p. 1.

line $P'P$ is of degree $2\times1\times1\times5-2\times1-2\times1$, or 6. And the two lines, a and b, double on the quintic, are triple on this sextic surface. Hence the images of the plane sections are sextic curves having in common two triple points where the lines a and b meet π_0, and thirteen simple points where the transversals meet it. If we did not know the number of the latter, we could infer it to be thirteen, since two sextics of the system can have only five variable intersections. As mentioned above, eleven of these transversals determine the other two, and there is a pencil of surfaces having the two double lines and the thirteen transversals. But the ruled sextic determined as above by any plane section of a quintic of the pencil meets the other quintics also in plane quintic curves. Thus we have ∞^3 and not ∞^4 such ruled sextics. Hence we have the following theorem.

THEOREM 1. *Of the ∞^4 plane sextics having in common two triple points and eleven simple points there are ∞^3 (instead of ∞^2) that pass also through two certain points determined by the given thirteen.*

We see how this determination is effected when we consider the images of the double lines. From each point of one double line we must draw the two tangents to the surface, one in each sheet, that meet the other. Hence the two images of a point of a double line lie on a line through the point in which the other double line meets π_0. Let the double lines a and b meet π_0 at B and A respectively. Since any plane through a double line is tangent to the surface where the residual cubic meets the double line, the image of a has a double point at B and a triple point at A, and is met by any line through B in three points and by any line through A in two (associated) points. Thus the image of a double line is a quintic which has a double point where the double line itself meets π_0, and a triple point where the other double line meets it, and which contains the points of intersection of the thirteen transversals with π_0. But a quintic that has a fixed triple point and a fixed double point is fully deter-

mined by 11 other simple points; and two such quintics have just two more intersections, which are the points we seek. This gives the following simple but striking result. In a plane, π_0, take two points A and B and 11 other points C. Let the two remaining intersections of the two quintics both of which contain the 11 points C, and which have respecitvely a triple point at A and a double point at B, and vice versa, be P and Q. Then if any two skew lines neither of which lies in π_0, be drawn through A and B, there will be a pencil of quintic surfaces having the two skew lines for double lines and containing the 13 transversals determined by the points C and P and Q.

The configuration in π_0 may be simplified by a quadratic transformation. Taking for fundamental triangle the points A and B in which the double lines meet the plane, and C, where any one of the transversals meets it, we see immediately that the sextics which are the images of the plane sections become quintics having double points at A and B, and passing through the twelve points that correspond to the points in which the other twelve transversals meet π_0. The images of the double lines become quartics, one having a double point at A and a simple point at B, and the other a node at B and a simple point at A, and both passing through the twelve new points. C goes into the line AB which is now the image of the line on the surface that originally corresponded to C. Evidently the point C can be chosen in thirteen ways. That is, the original plane system of sextics can be reduced to the system of quintics having in common two double points and twelve simple points in thirteen ways. This was noted by Clebsch. Referring now to the new system let l be the line on the surface whose image is AB. To sections of the surface by planes through l correspond quintics each composed of the line AB and a variable quartic through A and B and the other twelve base points. These quartics have only two variable intersections with AB, since l meets both double lines of the surface. It was seen above that the image of a double line is a quartic, having a

double point at A (or B) and a simple point at B (or A), and passing through the twelve simple base points. Hence the images of the double lines are special curves of the pencil of quartics that correspond to the plane sections through l; and the fact that they have double points at A and B shows that any two curves of the pencil have two intersections at each of these points. This may be restated as follows.

THEOREM 2. *The fourteen fundamental points are the base points of a pencil of quartics that have a common tangent at two of them.*

On the image of a double line the points are paired, since to a point on a double line must correspond two points in the plane. Thus the quintic corresponding to any plane section meets the image of a double line in two (associated) points aside from the fundamental points. Two quartics of the pencil are tangent to AB. The corresponding plane quartics on the surface are tangent to l where it is tangent to the parabolic curve, and their planes are doubly stationary, that is, they are stationary planes in the developable of the stationary tangent planes. Clebsch remarks that there are twenty-five quartics of the plane pencil that have a node aside from A or B. The number would seem to be twenty-three, since the two special quartics of the pencil (the images of the double lines) count for two each. That is, $3(n-1)^2-4=23$ for $n=4$. Thus if we consider a pencil of cubics that have a common tangent there are only ten $(3\times 2^2-2)$ that have a node aside from the one whose node is at the point of tangency. Thus the plane sections through l (and hence any transversal) are in general doubly tangent to the surface, but twenty-three of them are triply tangent; and evidently two, that is, the planes determined by the transversal and the double lines, cut the surface in conics, and are quadruply tangent.

In the above the plane system has been derived by geometrical projection and quadratic transformation. If we

start with the statement that the plane system consists of the quintics having in common two double points, A and B, and twelve simple points, we immediately see that there must be a relation among the base points, for otherwise they would determine only ∞^2 quintics. We may, however, easily find this relation by means of the theory of rational surfaces as developed by Caporali.* It tells us that the complete image of the double lines is a curve of degree eight which must consist of the two quartics described above. Considering them in connection with the pencil of quartics, the images of the plane sections through l, we find as before the relation connecting the base points.

Some other properties of the surface will now be developed. Let a be the double line whose image is the quartic with double point at A. To the quintic composed of this quartic and the line AB corresponds the section of the surface whose plane contains a and l. Therefore the point A must correspond to the conic cut from the surface by that plane. All directions at A correspond to points on this conic. The directions of the tangents of the image of a correspond to the points at which the conic meets a. To the direction AB corresponds the intersection of the conic with l which does not take place on the other double line, b. All sections of the surface by planes through l, including the conic in question, pass through the point where l meets b; and corresponding to that point we have at A the common direction of the pencil of quartics which are the images of those sections. The image of b has at A this same direction. Since b is a double line, its intersection with l corresponds to another point in the plane, which must lie on AB, and which is the fourth point in which the image of b meets AB, in addition to the double point at B and the simple intersection at A. Thus this pair of associated points on the image of b lie on a line, BA, through the double point B. In an exactly similar

* *Sistemi Triplimente Infiniti*, Collectanea Mathematica, Hoepli, 1881.

way all directions at B correspond to points on the conic whose plane contains l and the double line b.

It is a distinctive feature of this correspondence that any ray through the double point, A or B, of the quartic which is the image of a double line meets the curve in two associated points, that is, points which correspond to a single point of the surface. For the image of b and any line through A make up a quintic of the system, the corresponding section of the surface being a plane cubic whose plane contains b. This plane meets a in a point whose two images must lie on the line through A. Through any point of the surface there are two such plane cubics corresponding to rays through A and B in the plane. The points of a pair corresponding to a point of the double curve are such that the net of fundamental curves that pass through one pass also through the other. If two such points come together they are the image of a pinch point of the surface. Such points lie on the Jacobian of any net of fundamental curves. In our case, since we can draw six tangents to a uninodal quartic from its double point, there are six pinch points on each of the double lines. Again, from A, which is a simple point on the image of b, eight tangents can be drawn to that quartic. To these tangents correspond plane cubics on the surface which are tangent to b where it is tangent to the parabolic curve, and whose planes are stationary planes in the developable of the stationary tangent planes. There are of course eight such sections that contain a. To each of the twelve simple base points corresponds a line that meets both double lines. To a ray through A (or B) and one of these twelve points corresponds the conic whose plane contains b (or a) and the line corresponding to that base point. Thus there are twenty-six such conics, including the two whose planes pass through l. To the line joining any two simple base points, or to the conic through A and B and three simple base points, corresponds a skew cubic. There are thus $66+220$, or 286 of these. If two simple base points lie on a line through A the corresponding lines on the surface will meet on a, and

their plane, which contains b, meets the surface in a residual line, whose image is the ray joining A to the base points. That is, two of the twenty-six conics just mentioned have become line pairs. Clebsch has pointed out that the ∞^3 hyperboloids which contain the double lines meet the surface in residual skew sextics whose images are the conics through A and B. The ∞^2 hyperboloids that contain also the line l intersect the surface in residual skew quintics that have for their images the line AB and the ∞^2 lines of the plane.

The genus of a curve of order n on the surface does not exceed the greatest integer in $(2n^2+2n+33)/20$. Let m be the order of the corresponding curve in the plane, and let it have points of multiplicity $\alpha_1, \alpha_2, \cdots, \alpha_{12}$ at the simple base points, and of multiplicity β_1 and β_2 at A and B. Then we have $5m-[\Sigma\alpha+2\Sigma\beta]=n$. Letting $\Sigma\alpha+2\Sigma\beta=k$, we have $m=(n+k)/5$. Then the genus of the plane curve, which is the same as that of the corresponding curve on the surface, is

$$\frac{(n+k-5)(n+k-10)}{50}-\frac{1}{2}[\sum\alpha(\alpha-1)+\sum\beta(\beta-1)].$$

Differentiating with respect to the twelve α's and two β's, we have fourteen equations from which it follows easily that for a maximum genus $k=4n+10$, the α's are all equal to $(n+3)/5$, and $\beta_1=\beta_2=(4n+7)/10$. For these values the expression for the genus becomes $(2n^2+2n+33)/20$.

The characteristic numbers of the tangent cone are obtained without difficulty by the method applicable to rational surfaces, or by the general formulas of Salmon and Plücker. For the sake of comparison with the forms of the surface to be treated later we may note that the order of the tangent cone is 16. Its class, or class of the surface, is 34. Its genus is 23; and it has 42 cuspidal edges, and 40 double edges.

In concluding this paper we may consider some special groupings of the base points. The general methods of

producing nodes on unicursal surfaces are to allow two or more base points to coincide, or to place on a line a number of base points equal to the order of the fundamental curves. Thus, if two simple base points coincide we get an ordinary conic node. If three become consecutive, but not collinear, the result is a binode, B_3, that reduces the class by three, etc. In our case the fourteen fundamental points are the common intersections of a pencil of quartics that have a common tangent at two of them. Therefore, when we have chosen these two, say A and B, and the directions thereat, we have only nine at our disposal. The remaining three which are thus determined may be called residual points. We can not put five of the nine on a line, nor any of them on the line AB. But we can put four of them on a line, and we then get on the surface a fourteenth line, m, which meets l and the lines corresponding to the four base points, but which does not meet the double lines. To sections of the surface by planes through m correspond a pencil of quartics having double points at A and B, and passing through the five remaining points (of the nine) and the three residual points. An arbitrary quartic of the pencil meets the line containing the collinear base points four times, and thus the corresponding plane is quadruply tangent to the surface. But thirteen quartics of the pencil have a third node and their corresponding planes are quintuply tangent. If these ten points were arbitrary there would be but one quartic having double points at two of them and passing through the other eight. The section that contains m and l meets the double lines where l does. Its image in the plane, therefore, in addition to the line containing the four base points, is the line AB, and a cubic which has the common directions at A and B, and passes through the other eight points; thus satisfying twelve conditions. If we now put four of the five points still at our disposal on a line we get a fifteenth line, n, on the surface, which meets l and m and four others, but not the double lines. Clebsch remarked that no line can meet more than five of the thirteen trans-

versals to the double lines. The residual section by the plane of l, m, n is a conic whose image in the plane is also a conic, which has the common directions at A and B and passes through the four other points, thus fulfilling eight conditions. Thus, if four points of such a set of fourteen are collinear, or if eight of them lie in sets of four on two lines, the remaining points are related in the manner just noted. If we arranged the nine points in a triangle, one at each vertex and two on each side, we should get three skew lines on the surface, each meeting l and four other lines, but not meeting the double lines.

THE UNIVERSITY OF CALIFORNIA

A NOTE ON THE RATIONAL PLANE QUARTIC CURVE WITH CUSPS OR UNDULATIONS*

BY J. H. NEELLEY

1. *Introduction.* In a recent paper† compound singularities of the rational plane quartic curve have been considered, but cusps and undulations were not incorporated in that article because of their widespread discussion by other writers. However, the errors or ambiguities in previous treatments of these two singularities are cleared up by this paper.

2. *A System of Cusp Invariants.* It is well known that all types of the rational plane quartic curve with simple singularities are given to within projection by plane sections of the Steiner Romische Surface S_3^4 of order three and class four. When referred to its tropes S_3^4 has the equation

$$(1) \qquad (x_0)^{1/2} \pm (x_1)^{1/2} \pm (x_2)^{1/2} \pm (x_3)^{1/2} = 0.$$

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*Presented to the Society, December 28, 1927.

†J. H. Neelley, *Compound singularities of the rational plane quartic curve*, American Journal of Mathematics, vol. 49 (1927), pp. 389-400.

If the intersection of the three double lines is taken as the unit point these lines have the equations

$$(2) \qquad \begin{cases} x_0 = x_1, & x_1 = x_2, & x_2 = x_0, \\ x_2 = x_3, & x_3 = x_0, & x_1 = x_3. \end{cases}$$

The ends of the double lines are pinch points of the surface with the coordinates

$$(1,1,0,0),\ (1,0,1,0),\ (1,0,0,1),\ (0,1,1,0),\ (0,1,0,1) \text{ and } (0,0,1,1),$$

and a plane on one of these points cuts out a curve with a cusp at the point. Let the coordinates of the cutting plane be $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ so if it is on one of the pinch points, say (0, 1, 1, 0), we have $\alpha_1+\alpha_2=0$. As there are six pinch points there are six such forms each of which vanishes for some one of the pinch points. Therefore, one of the six factors of the product $\prod^{6}(\alpha_i+\alpha_j)$ vanishes if the curve has a cusp. So this product may be used as an invariant which vanishes for a curve with a cusp. Letting σ_i represent the symmetric functions of the α's, this invariant may be written

$$(3) \qquad \sigma_1\sigma_2\sigma_3 - \sigma_3^2 - \sigma_1^2\sigma_4 = 0.*$$

If we consider the curve with two cusps as cut out by a plane on two pinch points there are two forms of the type $\alpha_i+\alpha_j$ which vanish. So all possible products of five of the forms vanish and this gives as the two-cusp additional condition

$$\sum^{6}[\prod^{5}(\alpha_i + \alpha_j)] = 0,$$

which in terms of the symmetric functions is

$$(4) \qquad 2\sigma_1\sigma_2^2 + \sigma_2\sigma_3 + \sigma_1^3\sigma_3 - \sigma_1\sigma_4 = 0.*$$

But in so doing we neglect the fact that the plane cannot pass through both ends of one of the double lines. That is, a proper curve is cut out only if two forms such as $\alpha_0+\alpha_1$

*This invariant was given by J. E. Rowe, Transactions of this Society, vol. 12 (1911), p. 306.

and $\alpha_2+\alpha_3$ do not both vanish. Imposing this condition on the α's the two-cusp invariant is found to be

$$\sigma_1\sigma_3 + \sigma_2^2 - 4\sigma_4 = 0. \tag{5}$$

Now let the plane $(\alpha x)=0$ be on three pinch points. Then three forms $\alpha_i+\alpha_j$ vanish simultaneously so that all products of four such forms vanish and we have

$$\sum^{15}[\Pi^{4}(\alpha_i + \alpha_j)] = 0,$$

or

$$\sigma_1\sigma_3 + 32\sigma_4 + \sigma_2^2 + 2\sigma_1^3\sigma_2 = 0.^{*} \tag{6}$$

But again let us consider only proper curves and impose that no two forms such as $\alpha_i+\alpha_j$ and $\alpha_k+\alpha_l$ shall vanish simultaneously. This gives as the additional condition for a curve with three cusps the very simple invariant

$$\sigma_2 = 0. \tag{7}$$

3. *Degenerate Forms.* Now we consider the possible degenerate forms cut from this surface. Such are either a conic and the square of a line or the square of a conic or the squares of two lines. The conic and the square of a line is gotten if the cutting plane passes through only two pinch points and these must be the ends of one of the double lines of the surface. That is, if $\alpha_i+\alpha_j$ vanishes $\alpha_k+\alpha_l$ vanishes also, whence $\sigma_1=0$ for such. And this is the condition that the plane pass through the unit point which is necessary for degenerate forms including a line. As two forms $\alpha_i+\alpha_j$ vanish simultaneously the condition (4) of the previous section is satisfied. So for a conic and line we have

$$\sigma_1 = \sigma_3 = 0, \tag{8}$$

as $\sigma_2\neq 0$, since the plane does not pass through three pinch points.

The square of a conic is cut from the surface if the plane passes through three pinch points which lie in one of the

* This invariant is given by J. E. Rowe, loc. cit.

reference planes. This means that condition (6) is satisfied and also $\sigma_2=\sigma_3=0$ due to condition (7) and the fact that the cutting plane is tangent to the surface. This gives $\sigma_4=0$ and so the square of a conic occurs if

$$\sigma_2 = \sigma_3 = \sigma_4 = 0. \tag{9}$$

Now let the plane pass through two pinch points so that condition (5) is satisfied and also make $\sigma_1=0$ so that we have a pair of lines. This makes

$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0, \tag{10}$$

from consideration of relations (5) and (6).

So we have the following complete system:

A curve with one cusp has	$\sigma_1\sigma_2\sigma_3-\sigma_3^2-\sigma_1^2\sigma_4=0.$
Two cusps necessitates also that	$\sigma_1\sigma_3+\sigma_2^2-4\sigma_4=0.$
Three cusps adds to these	$\sigma_2=0.$
A conic and square of a line if	$\sigma_1=\sigma_3=0.$
The square of a conic if	$\sigma_2=\sigma_3=\sigma_4=0.$
The squares of two lines if	$\sigma_1=\sigma_2=\sigma_3=\sigma_4=0.$

4. *The Curve with Undulations.* There is another part of the literature of the rational plane quartic curve which should be revised. We refer to the theory of the curve with undulations as developed from consideration of the fundamental sextic which arises if the curve is represented parametrically. That is, if three forms

$$x_i = (\alpha_i t)^4, \qquad (i = 0,1,2),$$

represent the curve, the pencil of a polar quartics $(\beta t)^4+\lambda(\gamma t)^4$ fixes uniquely a sextic $(\rho t)^6$ which is apolar to the members of the pencil. The curve has been discussed by means of the invariants and covariants of this sextic.* Its line sections are the second polars of $(\rho t)^6$ and the pure second polars envelope the conic K† which is the locus of points in the plane from which a pencil of lines will cut out a pencil of

*W. Stahl, *Über die rationale ebene Curve vierter Ordnung*, Journal für Mathematik, vol. 104 (1889), pp. 302-320.

†W. Stahl, *Über die rationale ebene Curve vierter Ordnung*, Journal für Mathematik, vol. 101 (1886), pp. 300-325.

binary quartics having the invariant I_2 zero. This conic K breaks up into two lines when the quartic curve has an undulation; hence the statement of some writers that the sextic $(\rho t)^6$ is inadmissable when the curve has an undulation.* This statement is seemingly supported by the fact that the roots of $(\rho t)^6 = 0$ are the parameters of the points of the quartic curve where the lines of the curve and the conic K at its corresponding points are incident. But let us examine the curve with an undulation. Calling the parameter of the undulation $t = 0$ and that of a point of inflection $t = \infty$, the join of these two points and the lines of the curve at them gives the following representation of the curve:

$$(11) \qquad \begin{cases} x_0 = a_0 t^4, \\ x_1 = 4b_1 t^3 + 6c_1 t^2 + 4d_1 t, \\ x_2 = 4d_2 t + e_2, \end{cases}$$

whence the fundamental quartics are

$$(12) \quad (\beta t)^4 \equiv b_1 t^2 + c_1 t, \quad (\gamma t)^4 \equiv b_1 d_2 t^4 + b_1 e_2 t^3 - d_1 e_2 t^2.$$

These quartics give

$$(13) \qquad (\rho t)^6 \equiv K t^6$$

which is very definitely the sixth power of the undulation parameter. This sextic (13) permits the recovery of but one line section and that is the undulation line of the curve. But this line is also a line of the degenerate conic K whose equation is of the form

$$(14) \qquad x_0(K_0 x_0 + K_1 x_1 + K_2 x_2) = 0.$$

So at $t = 0$, the point fixed by the root of $(\rho t)^6 = 0$, we have the line of the curve incident with a line of the conic K even in this case. The unusual situation here is that the curve cannot be recovered from the sextic. This is generally possible

* W. Stahl, loc. cit., vol. 104, p. 303 and p. 318; R. M. Winger, *Self-projective rational curves of the fourth and fifth orders*, American Journal of Mathematics, vol. 36 (1914), p. 64.

as any three independent second polars of the sextic may be used as a reference scheme to recover the curve.

Consider a curve with undulations at $t=0$ and $t=\infty$. The join of the undulations and the undulation lines of the curve represent the curve as follows:

$$(15)\qquad \begin{cases} x_0 = a_0 t^4, \\ x_1 = \quad 4b_1 t^3 + 6c_1 t^2 + 4d_1 t, \\ x_2 = \qquad\qquad\qquad e_2, \end{cases}$$

with

$$(16)\qquad (\beta t)^4 \equiv b_1 t^2 + c_1 t, \qquad (\gamma t)^4 \equiv b_1 t^3 - d_1 t,$$

which give

$$(17)\qquad (\rho t)^6 \equiv \lambda t^6 + \mu \tau^4,$$

or a pencil of sextics which includes the sixth powers of each undulation parameter. All second polars of (17) are lines of the pencil determined by the undulation lines of the curve and once more the curve cannot be recovered from the sextic as there are not three independent second polars. The conic K for this type curve has the equation

$$(18)\qquad c x_0 x_2 = 0,$$

which is the product of the undulation lines, the two possible pure second polars of (17).

The curve with three undulations is well known but to make this article complete and to remove at least one erroneous statement concerning it we will give its theory. Let $t=0,\ \infty$ and c_1/b_1, where neither c_1 nor b_1 are zero, be the undulation parameters and use the undulation lines as a triangle of reference. Then the curve is given by

$$(19)\qquad \begin{cases} x_0 = a_0 t^4, \\ x_1 = (b_1 t - c_1)^4, \\ x_2 = e_2, \end{cases}$$

with

$$(20)\qquad (\beta t)^4 \equiv b_1^2 t^3 - c_1^2 t, \qquad (\gamma t)^4 \equiv b_1 t^2 - c_1 t$$

which have the net of apolar sextics

$$\lambda t^6 + \mu(b_1 t - c_1\tau)^6 + \nu\tau^6. \tag{21}$$

This net includes the sixth powers of the three undulation parameters as members. So the curve may be recovered from any member of the net (21) which is a linear combination of each of these sixth powers. This is because there is a net of second polars of such a member and any three independent ones may be used to recover the curve. The equation of the conic K vanishes identically if the curve has three undulations. This is obviously necessary from the definition of K. That is, the pencil of lines determined by any two lines $(\lambda x)=0$, $(\mu x)=0$ cuts a pencil of quartics from the curve (19) for which the invariant I_2 vanishes. So the equation of K must be satisfied by any point $(x_0x_1x_2)$ of the plane. We state the results of this section as the following theorems.

THEOREM 1. *A curve with one undulation has as its fundamental sextic the sixth power of the undulation parameter and the curve cannot be recovered from the sextic.*

THEOREM 2. *A curve with two undulations has a pencil of fundamental sextics which includes the sixth powers of the undulation parameters and again the curve cannot be recovered from the sextics.*

THEOREM 3. *A curve with three undulations has a net of fundamental sextics which includes the sixth powers of the undulation parameters and the curve may be recovered from any member of the net which is a linear combination of all three sixth powers.*

CARNEGIE INSTITUTE OF TECHNOLOGY

THE BITANGENTS AT A POINT OF A DESMIC SURFACE

BY R. M. MATHEWS

A plane tangent to a surface of the fourth order cuts the surface in a curve of the fourth order with a singular point at the point of contact. Through this point there can then be drawn six lines tangent to the curve elsewhere. So through a point on a surface of the fourth order there are in general six bitangents.

The second points of contact of these bitangents lie on a conic which, for a desmic surface, degenerates into two lines. Thus we have two sets of three bitangents; numerous geometrical properties of one triple have been given by G. Humbert and others.* Concerning the second triple only some general properties that apply to both sets are known; no characterizing geometric properties have been published. This paper presents one.

For brevity in the equations we shall use the symbol $\overline{ij}$ to denote $(x_i^2 - x_j^2)$.

The pencil of desmic surfaces in variables (y) and parameter (x),

$$(1) \quad \begin{vmatrix} (y_0^2 - y_1^2)(y_2^2 - y_3^2), & (y_0^2 - y_2^2)(y_1^2 - y_3^2) \\ \overline{01}\cdot\overline{23} & , \quad \overline{02}\cdot\overline{13} \end{vmatrix} = 0,$$

determines a cubic complex, and the cubic cone of this complex of which an arbitrary point $P(x)$ is the vertex, cuts the plane $y_3 = 0$ in the cubic curve

$$(2) \quad \overline{03}y_1y_2(x_2y_1 - x_1y_2) + \overline{13}y_2y_0(x_0y_2 - x_2y_0) + \overline{23}y_0y_1(x_1y_0 - x_0y_1) = 0.$$

* Humbert, G., *Sur la surface desmique du quatrième ordre,* Journal de Mathématiques, (4), vol. 7 (1891), pp. 353–398.

Jessop, C. M., *Quartic Surfaces with Singular Points,* Cambridge, 1916.

Mathews, R. M., *Cubic curves and desmic surfaces,* Transactions of this Society, vol. 28 (1926), pp. 502–522, and ibid., vol. 30 (1928), pp. 19–23.

The tangent plane at $P(x)$ to the desmic surface through P is

$$(3)\quad \overline{13}\cdot\overline{23}\cdot\overline{12}x_0y_0 + \overline{23}\cdot\overline{03}\cdot\overline{20}x_1y_1 + \overline{03}\cdot\overline{13}\cdot\overline{01}x_2y_2 + \overline{01}\cdot\overline{12}\cdot\overline{20}x_3y_3 = 0,$$

and cuts $y_3=0$ in a line ABC with

$$(4)\quad \begin{cases} A: \overline{03}x_1x_2 : \overline{13}x_2x_0 : \overline{23}x_0x_1, \\ B: \overline{03}(x_1x_2 + x_0x_3) : \overline{13}(x_0x_2 + x_1x_3) : \overline{23}(x_0x_1 + x_2x_3), \\ C: \overline{03}(x_1x_2 - x_0x_3) : \overline{13}(x_0x_2 - x_1x_3) : \overline{23}(x_0x_1 - x_2x_3). \end{cases}$$

The points A, B, C are on the cubic curve (2) and the lines PA, PB, PC are the known bitangents.*

The cubic curve (2) has for Hessian curve the cubic

$$(5)\quad \begin{vmatrix} \overline{23}x_1y_1 - \overline{13}x_2y_2, & \overline{23}(x_1y_0 - x_0y_1), & \overline{13}(x_0y_2 - x_2y_0) \\ \overline{23}(x_1y_0 - x_0y_1), & \overline{03}x_2y_2 - \overline{23}x_0y_0, & \overline{03}(x_2y_1 - x_1y_2) \\ \overline{13}(x_0y_2 - x_2y_0), & \overline{03}(x_2y_1 - x_1y_2), & \overline{13}x_0y_0 - \overline{03}x_1y_1 \end{vmatrix} = 0.$$

To write the coordinates of an arbitrary point Q on the line ABC, it is convenient to find $\overline{A}$ as the harmonic conjugate of A with respect to B and C, namely

$$(6)\qquad \overline{A} : \overline{03}x_0x_3 : \overline{13}x_1x_3 : \overline{23}x_2x_3$$

and then to write the coordinates for

$$Q \equiv \rho A + \sigma\overline{A}.$$

On solving $A\overline{A}$ simultaneously with the Hessian (5), we find that the values of the ρ/σ ratio for the points of intersection are roots of the cubic

$$(7)\quad 2\rho^3x_0x_1x_2x_3 + \rho^2\sigma\sum x_0^2x_1^2 + 6\rho\sigma^2x_0x_1x_2x_3 + \sigma^3(\sum x_0^2x_1^2 - \sum x_0^4) = 0.$$

Next, the line PQ is an arbitrary line through P on the tangent plane at P. We write the coordinates of an arbitrary point R on PQ in the form

$$R \equiv \lambda Q + \mu P$$

* For these equations and results see Mathews, loc. cit.

and substitute in the equation of the desmic surface. The result is a fourth degree equation in the ratio λ/μ whose four roots give the intersections R of PQ with the surface. Since P is a double point on the intersection of the plane and surface this quartic equation reduces to the quadratic

$$
(8)\quad \left\{\begin{aligned}
&\lambda^2\{(z_0^2 - z_1^2)z_2^2 n - (z_0^2 - z_2^2)z_1^2 m)\}\\
&\quad + 2\lambda\mu\{[(z_0^2 - z_1^2)z_2x_2 + z_2^2(x_0z_0 - x_1z_1)]n\\
&\quad - [(z_0^2 - z_2^2)z_1x_1 + z_1^2(x_0z_0 - x_2z_2)m]\}\\
&\quad + \mu^2\{[(z_0^2 - z_1^2)\overline{23} + z_2^2\overline{01} + 4(x_0z_0 - x_1z_1)x_2z_2]n\\
&\quad - [(z_0^2 - z_2^2)\overline{13} + z_1^2\overline{02}\\
&\quad + 4(x_0z_0 - x_2z_2)x_1z_1]m\} = 0,
\end{aligned}\right.
$$

where $m = k\ \overline{02}\cdot\overline{13}$, $n = k\ \overline{01}\cdot\overline{23}$, ($k$ a factor of proportionality), and where the z's are the coordinates of Q.

Now, when PQ is a bitangent to the surface the two roots of this quadratic are equal. We set the discriminant equal to zero, substitute for the z's the coordinates of Q as

$$Q = \rho A + \sigma\overline{A}$$

and obtain an equation of the sixth degree in the ratio ρ/σ. Three of the roots are $\sigma = 0$ and $\rho/\sigma = \pm 1$ giving A, B, C, the points for the known bitangents, and leaving for the determination of the other three exactly equation (7) which is the condition that Q lie at the intersections of the line ABC with the Hessian cubic.

The cubic cone at P cuts an arbitrary plane, not through P, in a cubic curve which has a definite Hessian; when the points of the latter are joined to P we get a cubic cone which may be called the Hessian cone corresponding to the first. Thus we have the following theorem.

THEOREM. *The bitangents to a desmic surface at a point thereon lie in two triples; the lines of one triple lie on the cubic cone of the system proper to P, and of the other on the corresponding Hessian cone.*

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ON CONNECTED AND REGULAR POINT SETS*

BY R. L. WILDER

In a recent number of this Bulletin† G. T. Whyburn showed that if A and B are any two points of a connected and regular point set M, and K denotes the set of all points of M which separate‡ A and B in M, then $K+A+B$ is a closed and bounded point set. In the present paper I shall show that this theorem is susceptible of quite simple proof, and admits of two obvious generalizations which hold in space of n dimensions. The methods used in proving these generalizations are also employed to show that if N is a closed and bounded point set which lies in a connected subset of an open subset, F, of a connected and regular point set, M, then F contains a bounded, connected and regular set which contains N. These results are then applied to give certain theorems concerning continuous curves.

Use will be made of the notions *region of* M and *simple chain of regions* as introduced in my paper, *The non-existence of a certain type of regular point set*,§ as well as of Theorem 1 of that paper; these extend readily to n-dimensional space, if in the definition of *region of* M "circle" is replaced by "sphere." Furthermore, since the Borel property is employed I make note of the following lemma, the proof of which should be quite obvious.‖

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* Presented to the Society, April 7, 1928.

† G. T. Whyburn, *Concerning connected and regular point sets*, this Bulletin, vol. 33 (1927), pp. 685–689.

‡ If A, B and X are points of a connected set M, then X is said to separate A and B in M if $M-X$ is the sum of two mutually separated point sets which contain A and B respectively.

§ This Bulletin, vol. 33 (1927), pp. 439–446. This paper will be referred to hereafter as N. E. R.

‖ It will be understood hereafter without explicit statement that the results hold for sets imbedded in euclidean space of n dimensions.

LEMMA. *If M is a connected and regular point set, and N is a closed and bounded subset of M, and G is a collection of regions of M which cover N, then G contains a finite subset which covers N.*

(By virtue of the property of regularity, there exists, concentric with the sphere used in determining a region R of M, a sphere S such that *all* points of M interior to S are also points of R. When it is said, then, that a collection of regions, G, *covers* a subset N of M, it is meant that every point of N is within the sphere S which corresponds to some region of the collection G.)

DEFINITION. If N is a subset of a connected set M and P is a point of $M-N$, then P is said to *separate N in M* if $M-P$ is the sum of two mutually separated sets each of which contains at least one point of N.

THEOREM 1. *If N is a closed and bounded subset of a connected and regular point set M and K denotes the set of all points which separate N in M, then $K+N$ is a closed and bounded point set.*

PROOF. Suppose the set $K+N$ is not closed. Then there is a point P which is a limit point of this set and which does not belong to it. As N is closed, P is a limit point of K, but not of N. If x is any point of M, distinct from P in case P is a point of M, there exists a region of M which covers x and which neither contains P nor has P as a limit point. Let G denote the collection of all such regions. By the above lemma, G contains a finite set of regions, R_1, R_2, $\cdots$, R_n, such that every point of N is in at least one of these regions. For every i ($i=1, 2, \cdots, n$), let P_i be a point of N in R_i. For every two points P_i and P_{i+1} ($i=1, 2, \cdots, n-1$) there exists, by Theorem 1 of N. E. R., and by virtue of the fact that P does not separate these points in M,* a

* Use is made in this connection and in the proof of Theorem 2 below, of Lemma 2 of my paper *A characteristization of continuous curves by a property of their open subsets*, Fundamenta Mathematicae, vol. 11(1928), pp. 127–131.

simple chain of regions of G from P_i to P_{i+1}. The regions $R_1, R_2, \ldots, R_n$ together with the regions which go to make up these simple chains form a finite set, g, of regions of G, and if H denotes the set of all points of M contained in regions of g, then H is a connected and bounded set. As P is not a limit point of H there exist in the vicinity of P points of K which are not in H. But clearly such points cannot separate N in M since N is a subset of H. Thus the supposition that $K+N$ is not closed leads to a contradiction.

To show that $K+N$ is bounded, let G be any collection of regions of M covering N and proceed as above to establish the existence of a bounded connected set, H, which is a subset of M and contains N. As every point of K must lie in H, it is clear that $K+N$ is bounded.

DEFINITION. If A and B are any two distinct subsets of a connected set M, and X is a point of $M-(A+B)$, then X is said to *separate A from B in M* if $M-X$ is the sum of two mutually separated sets which contain A and B, respectively.

THEOREM 2. *If A and B are any two distinct subsets of a connected and regular point set M, such that $A+B$ is closed and bounded, and K denotes the set of all points which separate A from B in M, then $K+A+B$ is a closed and bounded set.*

INDICATION OF PROOF. Select P and G as in Theorem 1. As P does not separate A from B in M, there is a connected subset of $M-P$ which contains a point P_1 of A and a point P_2 of B. Hence from G can be selected a simple chain from P_1 to P_2.

The corollarys of Theorems 1 and 2 applied to continuous curves* are obvious.

As a direct consequence of Theorems 1 and 2 and of the fact that the difference of two closed sets is both an F_σ and a G_δ,† we have the following theorem.

* In this paper a continuous curve is considered as a closed, connected and regular point set not necessarily bounded.

† See F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914, p. 306. An F_σ is the sum of a denumerable set of closed sets and a G_δ is the set of points common to a denumerable set of open sets.

THEOREM 3. *Let M be a connected and regular point set. Then if N is any closed and bounded subset of M, the set of all points which separate N in M is both an F_σ and a G_δ: and if A and B are any two distinct subsets of M such that $A+B$ is closed and bounded, the set of all points which separate A from B in M is both an F_σ and a G_δ.**

By methods similar to those used in proving Theorem 1 and noting in addition the fact that a region in a connected and regular point set is itself a regular set (Theorem 2 of N. E. R.), we have the following result.

THEOREM 4. *If K is a closed subset of a connected and regular point set M and N is a closed and bounded subset of a connected subset of $M-K$, then N lies in a bounded, connected and regular subset of $M-K$ which has no limit point in K.*

The analog of Theorem 4 for continuous curves may be stated as follows.

THEOREM 5. *If N is a closed and bounded subset of an open subset, Q, of a continuous curve M, and N lies in some connected subset of Q, then N lies in a bounded continuous curve which is a subset of Q.*

PROOF. Hahn has shown† that if P is any point of M and r is any positive number, there exists a continuous curve $M(P, r)$ which is a subset of M, contains every point of M less than a certain distance d (dependent on r) from P, and is such that all of its points are at a distance less than r from P. The set of all points $\{x\}$ of M such that x is joined to P by a connected subset of M every point of which is at a distance less than d from P constitutes a region of M and

* In this connection it may be of interest to note that it has been shown that the set of all cut points of a continuous curve is an F_σ. See C. Zarankiewicz, *Sur les points de division dans les ensembles connexes*, Fundamenta Mathematicae, vol. 9 (1927), pp. 124–171, Theorem 17.

† H. Hahn, *Mengentheoretische Charakterisierung der stetigen Kurve*, Wiener Akademie Sitzungsberichte, vol. 123, Part IIa, pp. 2433–2489; see Theorem XXI, p. 2475. Although Hahn states his result for a bounded regular continuum, it is clear that it holds for any regular continuum.

will be denoted by $M'(P, r)$. Clearly $M'(P, r)$ is a subset of $M(P, r)$.

If P is any point of Q, let r be a positive number less than the distance* from P to $M-Q$. Then the sets $M(P, r)$ and $M'(P, r)$ are subsets of Q. By a method similar to that used in proving Theorem 1 it can be shown that there exists a connected set H' which consists of a finite number of the regions of type $M'(P, r)$ and contains N. The set H composed of the sets $M(P, r)$ associated with those sets $M'(P, r)$ which constitute H' is a bounded continuous curve lying in Q and containing all points of N.

Theorem 5 is a generalization of a result obtained by R. L. Moore† to the effect that if Q is an open subset of a continuous curve and A and B are two points which lie in a connected subset of Q, then A and B are joined by a simple continuous arc which lies wholly in Q. As I have shown elsewhere‡ that this property is sufficient that a continuum be a continuous curve, it follows that the property stated in Theorem 5 also serves to characterize a continuous curve.

As I have indicated in a recent paper,§ subsets of a point set M may be separated in M in different senses. For our present purposes we employ the following definitions.

DEFINITION. If N is a subset of a connected set M and P is a point of $M-N$, then P is said to *separate N in M in the weak sense* if there exist two points of N which do not lie in a connected subset of $M-P$.

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* That is, the greatest lower bound of all distances Px, where x is a point of $M-Q$.

† *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260, Theorem 1.

‡ *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), pp. 340–377, Theorem 18.

§ *A characterization of continuous curves by a property of their open subsets*, Fundamenta Mathematicae, vol. 11 (1928), pp. 127–131. The terminology "*separates* in the strong (or weak) sense" should not be confused with "*disconnects* in the strong (or weak) sense" as introduced by R. L. Moore in *Concerning the cut-points of continuous curves and of other closed and connected point-sets*, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106.

DEFINITION. If A and B are any two distinct subsets of a connected set M and X is a point of $M-(A+B)$, then X is said to *separate A from B in M in the weak sense* if $M-X$ contains no connected subset which contains points of both A and B.

As examples of cases where these two definitions of "separate" are satisfied, but where the preceding definitions* are not satisfied, consider the following examples.

Examples. Let M_n $(n=1, 2, 3, \cdots)$ denote the straight line interval joining the points $(0, 0)$ and $(1, 1/n)$. Also, denote the points $(1/2, 0)$, $(1, 0)$ and $(0, 0)$ by A, B, and X, respectively, and let $M=A+B+\sum_{n=1}^{\infty}M_n$.

The set M is connected, but the set $M-X$ contains no connected set containing A and B. Hence X separates A from B in the weak sense. However, there is no separation of $M-X$ into two mutually separated subsets containing A and B, respectively, and thus X does not separate A from B in M in the strong sense, that is, in the sense of the definition which immediately precedes Theorem 2.

If we let N denote the set of all points with rational coordinates in the interval $[0, 1]$ of the X axis, except $(0, 0)$, and define M_n and X as above, and let M now denote the set $N+\sum_{n=1}^{\infty}M_n$, it is easy to see that X separates N in M in the weak sense, but not in the strong sense, i.e., in the sense of the definition preceding Theorem 1.

On the basis of Lemma 2 of my paper *A characterization of continuous curves by a property of their open subsets*† we have the following extension of Theorems 1–5.

THEOREM 6. (1) *Theorems* 1, 2 *and* 3 *still hold true if "separate" be interpreted to mean "separate in the weak sense"*; (2) *Theorem* 4 *still holds true if the words "N is a closed and bounded subset of a connected subset of $M-K$" be replaced by "N is a closed and bounded subset of $M-K$ such that there is no separation of $M-K$ into two mutually separated sets each*

* That is, the definitions preceding Theorems 1 and 2, respectively.
† Loc. cit.

of which contains points of N;" and (3) *Theorem* 5 *still holds true if the words "N lies in some connected subset of Q" be replaced by "there is no separation of Q into two mutually separated sets each of which contains points of N."*

In conclusion we may note the following application of part (1) of Theorem 6 to the theory of irreducibly connected sets.*

THEOREM 7. *If the connected and regular point set M is irreducibly connected about a closed and bounded set N, then M is a bounded continuous curve.*

PROOF. Let P be any point of $M-N$. Then $M-P$ contains no connected subset which contains N, since M is irreducibly connected about N. That is, $M-N$ is the set of points of M which separate N in M in the weak sense, and accordingly by Theorem 6, part (1), the set $(M-N)+N=M$ is bounded and closed. Hence M is a bounded continuous curve.

It may be pointed out that Whyburn's Theorem 2† to the effect that if a connected and regular point set M is irreducibly connected between two of its points A and B, then M is a simple continuous arc from A to B, is a corollary of Theorem 7 above. For since by Theorem 7 such a set, M, is a continuous curve, A and B are the end points of a simple continuous arc, t, of M. It is clear, then, that $M \equiv t$.

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* A connected set M is said to be *irreducibly connected* about one of its subsets, N, if it has no proper connected subset which contains N. See H. M. Gehman, *Concerning irreducibly connected sets and irreducible continua*, Proceedings of the National Academy of Sciences, vol. 12 (1926), pp. 544–547.

† Loc. cit. I might say here that in establishing the first of those results concerning simple closed curves to which Professor Whyburn kindly calls attention in this connection, I found it necessary to prove as a lemma the definition of arc stated in his Theorem 2. I did not mention this in my abstract (this Bulletin, vol. 32 (1926), p. 123, abstract No. 15) and have not yet published the paper. However, the proof which I developed in that connection is quite different from that given by Professor Whyburn as well as from the proof indicated in the present paper.

A PROPERTY OF THE LEVEL LINES OF A REGION WITH A RECTIFIABLE BOUNDARY*

BY G. A. PFEIFFER

1. *Introduction.* Before stating the result of this paper let me recall that a *level line of a region*† in a plane is the locus of the equation $g(x, y; a, b) = c$, where $g(x, y; a, b)$ is the Green's function of the region which has the point (a, b) as its pole, and c is a positive constant. The set of all level lines of the region, with the point (a, b) fixed and c any positive constant, is called a *pencil of level lines* of the region, and the fixed point (a, b) is called the *pole* of that pencil. Any pencil Σ of level lines of a region which is in a plane and which has a connected boundary containing more than one point, is the image of the set of circles concentric with and interior to any circle K under any transformation Π which maps in a one-to-one and conformal way the interior of K on Σ, such that the pole of the pencil of level lines corresponds to the center of K; and, conversely, the image of the set of circles concentric with and interior to a circle K under any transformation Π which maps in a one-to-one and conformal way the interior of K on a planar region Σ is a pencil of level lines of Σ, which has the image under Π of the center of K as its pole. Thus, a pencil of level lines of Σ is a one-parameter set of simple closed curves, and the value of the parameter t of a level line G of such a pencil of level lines may be taken as the length of the radius of the circle to which G corresponds under Π. Taking K as the unit circle, then t varies between 0 and 1. The symbol $[G_t]$ denotes a pencil of level lines of the region

* Presented to the Society, May 7, 1927, as part of a paper of the title *Certain sequences of curves which approach a rectifiable boundary from within.*

† A region in a plane is a set of points in a plane such that there exists a planar neighborhood of each point of the set which contains only points of the set.

Σ such that the level line G_t corresponds under Π to the circle which is concentric with the unit circle and which has a radius of length t $(0<t<1)$.

Now, if $\Lambda(t)$ denotes the function of t defined for $0<t<1$ and such that $\Lambda(t_1)$ is the length of the level line G_{t_1} of the pencil of level lines $[G_t]$, $0<t<1$, then it is a result of a theorem of Hardy, referred to below, that $\Lambda(t)$ is an increasing continuous function of t for $0<t<1$; that is, if $0<t_1<t_2<1$, then $\Lambda(t_1)<\Lambda(t_2)$. It is a simple consequence of the formula for the length of an analytic transform of a rectifiable curve, which is derived in §2 below, that $\lim_{t\to 0}\Lambda(t)=0$ and nothing further is said about that; but much of the following proof is devoted to showing that if the boundary of Σ is a rectifiable simple closed curve, then $\lim_{t\to 1}\Lambda(t)$ is the length of the boundary of Σ.

The result of this paper is contained in the following two theorems. The theorems are closely connected and their proofs are combined in the demonstration which follows.

THEOREM. *The function $\Lambda(t)$ of t, which is defined in the interval $0\leqq t\leqq 1$, and which is such that $\Lambda(t_1)$, if $0<t_1<1$, is the length of the level line G_{t_1} of the pencil $[G_t]$, $0<t<1$, of level lines of the planar region Σ whose boundary is a rectifiable simple closed curve and such that $\Lambda(0)=0$ and $\Lambda(1)=$ the length of the boundary of Σ, is an increasing* continuous function of t in the (closed) interval $0\leqq t\leqq 1$.*

DEFINITION. An *approximating sequence of regions* of the region Σ is a sequence of regions $\{\Sigma_n\}$, $n=1, 2, 3, \cdots$, such that every limit point of each Σ_n is a point of Σ and every point of Σ belongs to all but a finite number of the regions Σ_n.†

* That is, if $0\leqq t_1<t_2\leqq 1$, then $\Lambda(t_1)<\Lambda(t_2)$.

† It follows readily from this definition that if the region Σ is bounded then an approximating sequence of regions $\{\Sigma_n\}$, $n=1, 2, 3, \cdots$, of the region Σ contains a subsequence of regions $\{\Sigma_{n_i}\}$, $i=1, 2, 3, \cdots$, which is such that (a) every limit point of any region Σ_{n_i} belongs to Σ and to the succeeding region $\Sigma_{n_{i+1}}$ and (b) every point of Σ is in all but a finite number of the regions Σ_{n_i}.

DEFINITION. An *approximating sequence of curves* of the region Σ is a sequence of curves $\{C_n\}$, $n=1, 2, 3, \cdots$, such that each curve C_n is the boundary of a region Σ_n of an approximating sequence of regions $\{\Sigma_n\}$, $n=1, 2, 3, \cdots$, of the region Σ.

THEOREM. *If the boundary of a planar region Σ is a simple closed curve which is rectifiable, then there exist approximating sequences of curves $\{C_n\}$, $n=1, 2, 3, \cdots$, of the region Σ such that the curves C_n are level lines of any given pencil of level lines of Σ and, if l_n denotes the length of the curve C_n, $l_n<l_{n+1}$ and $\lim_{n\to\infty} l_n$ is the length of the boundary of Σ.*

2. *The Length of an Analytic Transform of a Rectifiable Curve.* Let the function $w=f(z)$ be analytic in the interior, $i(K)$, of the unit circle, K, and map in a one-to-one way $i(K)$ on the region Σ of the theorem. Let C be a rectifiable curve in $i(K)$ and C' its image under the transformation $w=f(z)$. Then the length, l', of C' is

$$\lim_{n\to\infty} (|\Delta w_{n_1}| + |\Delta w_{n_2}| + \cdots + |\Delta w_{n_r}|),$$

where $\Delta w_{n_i}=f(z_{n_i})-f(z_{n_{i-1}})$, where z_{n_0}, z_{n_1}, z_{n_2}, $\cdots$, z_{n_r}, $z_{n_{r+1}}=z_0$ are points on C such that $|z_{n_i}-z_{n_{i-1}}|<\delta_n>0$, and where $\lim_{n\to\infty}\delta_n=0$.

If Δs_{n_i} is the length of the arc of C whose end points are z_{n_i} and $z_{n_{i-1}}$ and which does not contain as an inner point the point $z=z_{n_0}$, then

$$l' = \lim_{n\to\infty} \sum_{i=1}^{r} \left|\frac{\Delta w_{n_i}}{\Delta z_{n_i}}\right| \cdot \Delta s_{n_i}, \qquad \Delta z_{n_i} = z_{n_i} - z_{n_{i-1}};$$

and, because of the uniformity of the approach of $\Delta w/\Delta z$ to dw/dz along C, dw/dz being continuous on C, it follows that

$$l' = \lim_{n\to\infty} \sum_{i=1}^{r} \left(\lim_{z\to z_{n_i}} \left|\frac{\Delta w}{\Delta z}\right| \right) \Delta s_{n_i}.$$

Hence

$$l' = \int_C \left|\frac{dw}{dz}\right| ds.$$

3. *The Level Lines of a Polygonal Region.* Let $i(J)$ denote a region whose boundary is a simple polygon J, and let $w=f(z)$ be a function which is analytic in the interior $i(K)$ of the unit circle K and which maps in a one-to-one way $i(K)$ on $i(J)$. Then $w=f(z)$ is analytic at any point of the circle K whose image is not a vertex of J and if $w=f(a)$ is a vertex of J, then at any point of $i(K)$ different from $z=a$ in some neighborhood of $z=a$ the derivative of $w=f(z)$ is $(z-a)^{\mu}\,\lambda(z)$, where $\lambda(z)$ is analytic at $z=a$ and $-1<\mu<1$.* In fact, $\mu=\alpha/\pi-1$, where α is the measure in radians of the interior angle of the polygon J whose vertex is the point $w=f(a)$.

Let the point $w=f(a_i)$ be a vertex of the polygon J and U_{a_i} a neighborhood of $z=a_i$ such that at every point of $i(K)$ which is in U_{a_i} and different from $z=a_i, f'(z)=(z-a_i)^{\mu_i}\lambda_i(z)$, where $\lambda_i(z)$ is analytic at $z=a_i$ and $-1<\mu_i<1$. Further, let Γ_i denote a circular arc concentric with K, and contained in U_{a_i}, such that its mid-point $z=b_i$ is on the radius of K through $z=a_i$. Let $z=c_i$ be an end point of this arc. Then, if $-1<\mu_i<0$ and M_i is a bound of $|\lambda_i(z)|$ in U_{a_i},

$$\int_{\Gamma} |f'(z)|\,ds \leqq M_i \int_{\Gamma_i} (|z-a_i|)^{\mu_i} ds \leqq M_i \int_{\Gamma_i} (|z-b_i|)^{\mu_i} ds.$$

If η is an arbitrary positive number, then there exists a positive number ξ such that $|z-b_i|/(\widehat{zb_i}) \geqq 1-\eta$ if $\widehat{zb_i}<\xi$, where $\widehat{zb_i}$ denotes the length of the sub-arc of Γ_i whose end points are $z=z$ and $z=b_i$. Then

$$\int_{\Gamma_i} (|z-b_i|)^{\mu_i} ds \leqq (1-\eta)^{\mu_i} \int_{\Gamma_i} (\widehat{zb_i})^{\mu_i} ds$$

$$= 2(1-\eta)^{\mu_i} \int_0^{l_i/2} s^{\mu_i} ds = 2(1-\eta)^{\mu_i} \frac{1}{\mu_i+1} \cdot \frac{l^{\mu_i+1}}{2^{\mu_i+1}},$$

* Bieberbach, *Lehrbuch der Funktionentheorie*, vol. II, p. 34 and p. 37. Also Study, *Vorlesungen über ausgewählte Gegenstände der Geometrie*, Part II, p. 85.

where l_i is the length of Γ_i and if $\eta<1/2$, then

$$\int_{\Gamma_i}(|z-b_i|)^{\mu_i}ds<\frac{1}{2^{2\mu_i}(\mu_i+1)}l_i^{\mu_i+\eta}.$$

Again, if $0\leqq\mu_i<1$ and M_i is a bound of $|\lambda_i(z)|$ in U_{a_i}, we have

$$\int_{\Gamma_i}|f'(z)|\,ds\leqq M_i\int_{\Gamma_i}(|z-a_i|)^{\mu_i}ds\leqq M_i\int_{\Gamma_i}|c_i-a_i|\,ds$$

$$\leqq M_i\left(|a_i-b_i|+\frac{l_i}{2}\right)l_i,$$

where l_i is the length of the arc Γ_i. Hence if ϵ is any positive number there exists a positive number δ_i such that if Γ_i is any circular arc which is concentric with and either interior to or on the given circle K and contained in U_{a_i} and which has a length $l_i<\delta_i$, then

$$\int_{\Gamma_i}|f'(z)|\,ds<\epsilon.$$

Now, let σ_i denote an arc on the circle K which has $z=a_i$ as its mid-point and a length l_i which is less than δ_i and, further, such that no two arcs σ_i have a point in common and let R denote the set of all points which are interior to K and which do not belong to any sector bounded by the arc σ_i and the radii of K through its end points; also let $\overline{R}$ denote the set of points consisting of the points of R and of the boundary of R. Then $w=f'(z)$ is analytic in $\overline{R}$ and hence there exists a positive number δ such that $|f'(z_1)-f'(z_2)|<\epsilon$ if $z=z_1$ and $z=z_2$ are any two points in $\overline{R}$ such that $|z_1-z_2|<\delta$. Now, if d is a positive number less than the radius of each U_{a_i}, let K' denote a circle interior to and concentric with K and having a radius which differs from that of K by less than d and also less than δ. Then let τ_i denote any arc of K which contains no arc σ_i and whose end points are also end points of arcs σ_i and let σ_i' and τ_i' denote the arcs of K' which are composed of the points of intersection of the circle K'

and the radii of K through all the points of σ_i and all the points of τ_i respectively. It follows that

$$\int_{\tau_i'} |f'(z)|\,ds = \int_{\tau_i} \frac{r-d}{r}(|f'(z)| + \eta(z))ds,$$

where r is the radius of K and $|\eta(z)| < \epsilon$. If h denotes the length of the polygon J and h' the length of the transform of K' under the transformation $w=f(z)$ and n the number of vertices of J, then

$$h' = \sum_{i=1}^{n} \int_{\sigma_i'} |f'(z)|\,ds + \sum_{i=1}^{n} \int_{\tau_i'} |f'(z)|\,ds$$

$$< n\epsilon + \sum_{i=1}^{n} \int_{\tau_i} |f'(z)|\,ds + \frac{r-d}{r}\epsilon h,$$

and

$$h' > \sum_{i=1}^{n} \int_{\tau_i} |f'(z)|\,ds - \frac{d}{r}h - \frac{r-d}{r}\epsilon h.$$

Since

$$h - n\epsilon < \sum_{i=1}^{n} \int_{\tau_i} |f'(z)|\,ds < h,$$

it follows that

$$h - n\epsilon - \frac{d}{r}h - \frac{r-d}{r}\epsilon h < h' < n\epsilon + h + \frac{r-d}{r}\epsilon h.$$

Now if $d<\epsilon$, then

$$h - \left(n + \frac{h}{r} + h\right)\epsilon < h' < h + (n+h)\epsilon,$$

or

$$|h - |h'| < \left(n + \frac{h}{r} + h\right)\epsilon.$$

Hence if ρ is any positive number and d is sufficiently small then $|h-h'|<\rho$.

From this result follows immediately, as far as it concerns the $\lim_{n\to\infty} l_n$, the special case of the second theorem of the

paper in which the region Σ is the interior of a simple polygon in a plane.

4. *The Region Σ in General.* Let Σ be a region in the w-plane and let the boundary of Σ be a rectifiable simple closed curve, C. Then no level line of Σ has a length greater than the length of the boundary of Σ. For let there exist a level line of Σ, say G, which has a length, g, which is greater than the length, l, of C. It is assumed that $w=0$ is an interior point of G; no loss of generality follows from this assumption. Then there exists a function $w=f(z)$ which is analytic in the unit circle and which maps in a one-to-one and conformal way the interior of the unit circle on Σ such that $f(0)=0$ and $f'(0)=1$ and such that G is the image under the transformation $w=f(z)$ of a circle, H, concentric with the unit circle, K. Further, let $\{P_n\}$, $n=1, 2, 3, \cdots$, be a sequence of simple polygons inscribed in C which are such that $\lim_{n\to\infty} d_n=0$, where d_n is the length of a side of P_n which is not shorter than any other side of P_n, and such that $w=0$ is an interior point of each P_n. Then there exists a function $w=f_n(z)$ which maps the interior of the circle $|z|=1$ on the interior of P_n in a one-to-one and conformal way such that the point $w=0$ corresponds to the point $z=0$ and the derivative of the function $w=f_n(z)$ at $z=0$ is unity. By a theorem* of Carathéodory, and the fact that there is only one function which maps the interior of the unit circle on Σ in a one-to-one and conformal way such that $w=0$ corresponds to $z=0$ and the derivative of the mapping function at $z=0$ is unity, it follows that the sequence of functions $\{w=f_n(z)\}$, $n=1, 2, 3, \cdots$, approaches the function $w=f(z)$, $|z|<1$, uniformly on any closed set of points which is contained in the interior of the unit circle. Consequently the sequence of derivatives of the functions $w=f_n(z)$, $\{w=f_n'(z)\}$, $n=1, 2, 3, \cdots$, converges uniformly to the derivative of $w=f(z)$ on any closed set of points which is in the interior of the unit circle.

* See Mathematische Annalen, vol. 72 (1912), pp. 120–126. Also Bieberbach, *Lehrbuch der Funktionentheorie*, vol. II, pp. 12–15.

Now, let ν be a positive number less than $g-l$ and p a positive integer such that

$$|f'(z) - f_p'(z)| < \frac{\nu}{2\pi},$$

for z on H. Then

$$|f_p'(z)| = |f'(z)| + \eta(z),$$

where $|\eta(z)| < \nu/2\pi$ for z on H, and

$$\int_H |f_p'(z)|\, ds = \int_H |f'(z)|\, ds + \int_H \eta(z) ds.$$

But

$$\int_H |f'(z)|\, ds = g \text{ and } \int_H |f_p'(z)|\, ds$$

is the length of the image of H under the transformation $w=f_p(z)$, $|z|<1$. The latter image is a level line of the pencil of level lines of the polygonal region bounded by P_p, which has the point $w=0$ as its pole. If the length of this level line is denoted by g_p, then $g_p > g-\nu$ and hence $g_p > l$.

According to the result for polygonal regions which was obtained above, there exists a circle, H', with center $z=0$ and a radius of length less than unity but greater than the length of a radius of H such that the length, g_p', of the image of H' under the transformation $w=f_p(z)$ differs from the length, l_p, of P_p by an amount less than g_p-l. Since $l_p<l$ it follows that $g_p' < g_p$. But this result contradicts the fact that by a theorem* of Hardy in connection with the formula for the length of an analytic transform of a rectifiable curve, which is given above. It follows that $g_p' > g_p$.

* See Proceedings of the London Mathematical Society, (2), vol. 14 (1915), pp. 269–277. Also Landau, *Ergebnisse der Funktionentheorie*, p. 85. The theorem is that if $w=f(z)$ is analytic and not constant for $|z|<R$, then $\int_0^{2\pi} |f(re^{i\theta})|\, d\theta$, $z=re^{i\theta}$ $0 \leqq \theta \leqq 2\pi$ and $0<r<R$, is a continuous increasing function of r for $0<r<R$.

Only a special case of this theorem is used above. The functions concerned are only those which map in a one-to-one and conformal way the interior of the unit circle on the interior of a simple polygon.

Thus the length of any level line of Σ is not greater than the length of the boundary of Σ. The theorem of Hardy then implies that the length of any level line of $\sum$ is less than the length of the boundary of Σ.

That any sequence of level lines $\{G_{t_n}\}$, $t_n<t_{n+1}$, $n=1, 2, 3, \cdots$, and $\lim_{n\to\infty} t_n=1$, which belong to any given pencil of level lines $[G_t]$, $0<t<1$, of level lines of any planar region Σ whose boundary is connected and contains more than one point is an approximating sequence of curves of Σ follows easily from the one-to-one conformal mapping of the interior of the unit circle on Σ, which determines the sequence of level lines $\{G_{t_n}\}$, $t_n<t_{n+1}$, $n=1, 2, 3, \cdots$, as the image of a sequence of circles, $\{H_n\}$, $n=1, 2, 3, \cdots$, which are concentric with and interior to the unit circle and such that $\lim_{n\to\infty} r_n=1$, where r_n is the length of the radius of the circle H_n. Evidently, the level line G_{t_n} is in the interior of the level line $G_{t_{n+1}}$.

Now, if the boundary of Σ is a rectifiable simple closed curve of length l, it follows readily from certain known results* that if l_n is the length of the rectifiable curve C_n of the approximating sequence of curves $\{C_n\}$, $n=1, 2, 3, \cdots$, of the region Σ and if $\lim_{n\to\infty} l_n$ exists, then we have $\lim_{n\to\infty} l_n \geqq l$. Hence, with what has preceded, if l_{t_n} is the length of the level line G_{t_n} of the sequence of level lines $\{G_{t_n}\}$, $t_n<t_{n+1}$, $n=1, 2, 3, \cdots$, $\lim_{n\to\infty} t_n=1$, then $l_{t_n}<l$ and, since by the theorem of Hardy $l_{t_n}<l_{t_{n+1}}$, $n=1, 2, 3, \cdots$, it follows that $\lim_{n\to\infty} l_{t_n}=l$. Thus $\lim_{t\to 1} \Lambda(t)=l$ and the sequence of level lines $\{G_{t_n}\}$, $t_n<t_{n+1}$, $n=1, 2, 3, \cdots$, $\lim_{n\to\infty} t_n=1$, is an approximating sequence of curves of Σ as specified in the second theorem.

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* In particular, Theorem V, p. 519 of Hahn, *Theorie der reellen Funktionen*, vol. I.

CORRESPONDENCE OF GAUSS AND GERLING

Briefwechsel zwischen Carl Friedrich Gauss und Christian Ludwig Gerling. By Dr. Clemens Schaeffer. *Herausgegeben im Auftrage der Gesellschaft zur Beförderung der gesamten Naturwissenschaften zu Marburg.* Berlin, Otto Elsner, 1927. xx+820 pp. Price 36 marks; bound, 40 marks.

It is not too much to say that Germany sets a standard not reached by any other country in the recognition, through publications of this kind, of the work of a nation's great scientists. More than a half a century ago there appeared six volumes of the correspondence between Gauss and Schumacher; a quarter of a century ago there were added two volumes of letters exchanged between him and others; and beside these there has been published a considerable amount of correspondence between him and both Bessel and Humboldt. There now appears the present work of more than eight hundred pages, giving 388 letters dating from 1810 to 1854, of which 163 are those written by Gauss and 225 by Gerling.

Such extensive publications as these relating to the friendly intercourse between the great Göttingen scientist and his fellow workers are not often possible. Such a body of material is not available, even in the case of men of highest standing,—men whose letters would naturally be preserved. Some, like Cayley, were not prolific letter writers; while others, like Sylvester, wrote apparently for the love of writing. It was not, however, because Gauss was accustomed to correspond so freely with his friends that his letters have been so freely made known to the world, nor was it merely because Germany or a German learned society desired to honor his memory. It is in a large measure due to the fact that German readers are numerous enough to purchase volumes of this kind and thereby render their publication possible. Such source material is invaluable to the historian, but there are few historians to use it; the support of the publication must have come from the large body of cultured scientists who find in the past a stimulus to their work for the future. In our own country, with all its wealth, such publications seem to be almost impossible. Perhaps it is because letter writing is with us a lost art; more probably it is because reading is such.

Gerling was professor of mathematics, physics, and astronomy in the the University of Marburg from 1817 to 1864. He was repeatedly offered positions elsewhere, but for nearly a half a century he continued to fill the chair for which he had been chosen in his early years. He was eleven years younger than Gauss and an acquaintance formed in Göttingen had been maintained until the latter's death in 1855.

Professor Schaeffer, the editor, has shown himself admirably adapted to the work. His tastes are those of both Gauss and Gerling. He is professor of physics at Breslau, and hence his field of major interests was at least one of the fields which each writer cultivated with a success that the world recognizes. As editor he has been painstaking in giving the letters as they would have been written today, with modern spelling and punctu-

ation, and in supplying a large number of footnotes of a biographical and bibliographical nature. While many readers may wish that the text had been given precisely as the original letters were written, it should be said that all alterations from those mentioned are made in brackets so that the reader is sure of the text as Gauss or Gerling would have prepared it at the present time.

It is, of course, impossible to call attention to all points of historical interest in such a collection, but some idea of the scientific range may be obtained from a few references. Gerling told Gauss as late as Nov. 22, 1818 (see p. 182) that the construction of the 17-gon as given in the *Disquisitiones Arithmeticae* was not clear; indeed, that it was merely suggestive; and he asked for a complete proof. This Gauss gave (letter of Jan. 6, 1819, p. 185) in the form which is now familiar, together with a historical note (p. 188) concerning his discovery of the method. The work of Gauss and Weber on the electric telegraph is well known, and in his letter of Aug. 26, 1835, he goes into the subject of his later improvements (p. 447). Gerling's reply (p. 451) shows how much interest was being aroused at that time in the new invention.

We usually think of Gauss as a great mathematical genius and as a great astronomer, forgetting that he ranked relatively as high as a geodecist. Those who are interested in this phase of his work will find a large amount of material in this correspondence, not merely with respect to triangulations and accuracy of observation, but as to the advanced phases of the subject and as to the improvement of instruments and the application of the method of least squares to geodetic work.

As to his pure mathematical interests, there are numerous references to his early work on critical points of a triangle (p. 340), the question of the angle (p. 195), and the various theories of parallels, although these are generally in the nature of obiter dicta rather than of proof. Speaking of Legendre's treatment of parallels, for example, he remarks "das für mich gar keine Beweiskraft in seinem Schluss liegt," and he proceeds succinctly to give his reasons, and similarly in various other matters of a similar kind.

Naturally the work should and will find place in all mathematical and reference libraries of importance.

DAVID EUGENE SMITH

EXTERIOR BALLISTICS

In the March-April number of this Bulletin, (vol. 34, pp. 229–232) appears a review of Moulton's *Exterior Ballistics*. This has certain features not closely connected with the book under review which appear in need of comment, and in one case correction.

The first of these is the statement that "the student and the professor of mathematics have at their disposal no book from which a trajectory can be actually computed" the reason being* that the necessary "tables are regarded as confidential information by the U. S. Army."

There are, however, at least three places in which these tables are accessible to any one interested. These are *The Method of Numerical Integration in Exterior Ballistics* by Dunham Jackson, the *Exterior Ballistic Tables* of 1924, and Herrmann's *Exterior Ballistics*. The first two of these are War Department documents, but are not confidential. The last is published by the Naval Institute.

The second feature concerns the value of the method of numerical integration. It has become clear from a correspondence with the reviewer that he regards this method as a useless refinement in the present stage of experimental ballistics. He agrees, however, it has the merit of not adding further errors to those of experimental origin. Its use in Moulton's book to the exclusion of other methods accords with the avowed object of that book of remaining applicable over a considerable period of experimental advance. As regards actual computations, that advance is even now taking place. Many trajectories are now computed at Aberdeen using more recent data on the resistance function than those of the G table. The advantage of numerical integration here becomes evident. The computer is merely given a new table of the resistance function and he proceeds as before. The physicist does not have to ask the mathematician "can you integrate these new differential equations that my experiments give?"

The fact that numerical integration is extremely laborious is not brought out in the review, although of course well known to the reviewer. The individual steps are easy to perform and easy to understand but are very numerous. It would appear that this fact should be stressed in any general estimate of the value of the method, as it constitutes the outstanding objection to its use.

L. S. DEDERICK

ABERDEEN PROVING GROUND, Md.

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* Professor Rowe desires to add that, in his opinion, the theory and the tables should be given in the same book. THE EDITORS.

SHORTER NOTICES

The Geometry of René Descartes, translated from the French and Latin by David Eugene Smith and Marcia L. Latham. With a facsimile of the first edition, 1637. Chicago and London, The Open Court Publishing Company, 1925. xiii+246 pp., with portrait. Price, $4.

La Géométrie de René Descartes. Nouvelle Edition. (Avec portrait de Descartes d'après Frans Hals.) Paris, J. Hermann, 1927. Price, 21 francs.

It is indeed a healthy sign of the popular interest in the history of mathematics that so many publishers in European countries, and in America, are occupied with the reproduction of the classics of science. One can find few works more worthy of reproduction than this fundamental achievement of Descartes. In a very real sense modern mathematics begins with his analytic geometry. Also in a very real sense Descartes is the discoverer rather than the inventor of the work. We now can see that this work is the product of the varied developments of mathematics made in Europe before Descartes, particularly in Italy, in France, and in England, development of ideas which were based upon the work of the Greeks, the Hindus, and the Arabs. The contemporary and even more modern work by Fermat illustrates the fact that the ideas were "in the air." Fortunately both were Frenchmen.

The French edition of Descartes' Geometry is the second publication of this work in this form, although this fact is not stated in the publication. The first edition appeared in 1886, issued by the same publisher. The French edition unfortunately modernizes the terminology.

Two German editions have also appeared, translated by Ludwig Schlesinger. The first appeared in 1894 and sold for three marks sixty pfennigs which is again the price for the new edition of 1923, a noteworthy achievement. The French edition sells at 21 francs.

The American edition is far superior to the others mentioned in that it gives an exact facsimile of the text, with translation and notes. The price, four dollars, is just about five times that of a French or German edition.

The notes are, in general, adequate. The vital point that Descartes studiously avoids negative abscissas and ordinates does not receive treatment. Some of the notes, for example, note 49 on page 33, are actually misleading on this point. In recent publications Wieleitner has stressed this late complete recognition of the negative. Tropfke (*Geschichte der Elementar Mathematik*, vol. 6, Leipzig, 1924, p. 109) asserts that only with Newton in 1704 do the four quadrants receive equal recognition, and that most texts of the eighteenth century did not grasp it.

The English version can be highly commended to all students of mathematics, a fine rendition of a classic which will even today be a source of inspiration to lovers of mathematical science.

L. C. KARPINSKI

Mathematische Formelsammlung. By O. Th. Bürklen, revised by F. Ringleb. Berlin and Leipzig, Walter de Gruyter, 1927. 241 pp.

This little volume of the Sammlung Göschen contains a very complete list of the important formulas and theorems of what the mathematician would call elementary mathematics. The subjects included are arithmetic, algebra, elementary theory of numbers, plane and solid geometry, plane and spherical trigonometry with applications to geography and astronomy, plane and solid analytic geometry, differential and integral calculus, differential geometry, and differential equations.

The table of contents and a brief index make it very easy to locate a formula or theorem and the typography is excellent. The book may be recommended as a convenient source for those of us who hesitate to trust our memories for formulas and theorems which we seldom use.

Under the heading "Literatur" there occurs only one citation, namely, the new edition of Pascal's Repertorium by Salkowski and Timerding, which is likely to form the basis of any other collection of mathematical formules.

W. R. LONGLEY

Les Bases de la Géométrie et de la Physique. L'Invariance de L'Espace Euclidien. Par Clément Laurès. Paris, Blanchard, 1928. 125 pp.

According to a statement on page 1, the physicist will say that the remarkable usefulness of euclidean geometry in physics is due to the fact that it investigates the relations of invariable solid bodies, and that measurements made in accordance with it reveal veritable constants of nature; while the mathematician will affirm that there is an infinitude of systems of geometry all equally possible and that the basis of choice among them rests on considerations of convenience, and not on logical necessity. To M. Laurès, the latter answer is unsatisfactory; and that of the relativists is still more so. If the parallel postulate of Euclid is indemonstrable, then (according to the author, p. 4) geometry has no solid foundation. To show how to demonstrate this postulate is the avowed purpose of the book—a purpose which the author (p. 5) "presents with confidence to the enlightened public." He says (p. 5): "The indemonstrability of this postulate is the first article of the contemporary *Credo* of science. The reader will see how this Credo is false on this point." "The non-euclidean geometries were one of the most stupid inventions of the nineteenth century." "The human reason is sick." And here the author undertakes to administer that intellectual remedy which shall restore it to health. With this restoration he would also remove the "stupidity" of relativity from modern science. "Whatever contains truth," he says on page 5, "can be very simply explained by means of invariant euclidean space." It is improbable that any mathematical reader will now be seriously interested in an argument whose avowed purpose is to demonstrate the parallel postulate of Euclid.

R. D. CARMICHAEL

Fonctions Entières et Fonctions Méromorphes d'une Variable. By G. Valiron. Mémorial des Sciences Mathématiques, Fascicule II. Paris, Gauthier-Villars, 1925. 56 pp.

Sur les Fonctions Hypergéométriques de Plusieurs Variables. Les Polynomes d'Hermite et autres Fonctions Sphériques dans l'Hyperspace. By Paul Appell. Mémorial des Sciences Mathématiques, Fascicule III. Paris, Gauthier-Villars, 1925. 75 pp.

So brief a time has elapsed since the appearance of Valiron's former work, *Lectures on the General Theory of Integral Functions*, that one naturally expects the present small volume to follow essentially the development given in the former book. The condensation of material in order to confine the subject to fifty pages is most noticeable. On this account the beginner in this field will probably prefer the older book, where the pace is more leisurely and the exposition more detailed. The bibliography is more complete in the later treatment. The author's own fertility almost caused him embarrassment, since he just exhausted the alphabet in listing his own titles.

In the work on the hypergeometric functions of several variables Appell has brought together a brief account of the work, extending over a number of years, of a number of mathematicians, of whom he himself is very conspicuous. The work consists of generalizations in rather natural directions. Guided by the celebrated series of Gauss, four different series in two variables and four to five parameters are constructed. For the functions so defined properties analogous to those of the hypergeometric function are developed. One expects to find nothing especially novel or interesting, and his expectations are confirmed. It seems to the reviewer that a number of celebrated mathematicians have deprived lesser ones of much fine material for theses. It is, of course, useful to have an account of what has been done in the field, even though it has limited significance, and it gives gratification to have it done so ably.

K. P. Williams

Exercices de Calcul Différentiel et Intégral. By Abbé Potron. Volume I. Paris, Hermann, 1926. xviii+332 pp.

This book is a typical representative of the set of French "Exercises" or "Collections of problems." The final goal here is to train the students for the technically difficult examination for the "Certificate in Differential and Integral Calculus."

Accordingly, the problems which have been given at such examinations for a number of years in Paris and in other cities, are "analysed" here, or decomposed into their simplest elements, where the solution is reduced to a mere question of routine computation. The "synthesis," namely the problems as they have been actually given and their solutions, are promised in the second volume.

This method perhaps is well fitted for the particular purpose as mentioned above, but it is at least questionable as to its general educational value. However, American teachers will find many a good example to be

assigned to students for their home-work, among the total of 1080 problems gathered in this book.

The problems are classified into six parts: I. Derivatives, differentials, geometric applications. II. Integral calculus. III. Differential equations (ordinary). IV. Partial differential equations. V. Functions of a complex variable. VI. Supplements. Each part is subdivided into smaller chapters, and the latter in turn into paragraphs. Each paragraph is supplied with short outlines of the corresponding parts of the theory, which not always are above reproach, particularly when the attempt is made to be rigorous.

Most completely treated are the applications of calculus to geometry, especially those which require formal computations. Some other topics, however, are treated quite inadequately: they might have been omitted altogether without any loss; as for instance, ruled surfaces (2 problems), contact of plane curves (1), improper integrals (3), elliptic and hyperelliptic integrals (6), transformations of surfaces (3), trigonometric series (4), calculus of variations (6) and so on.

The order of the problems is not always the best possible; some problems are repeated at different places. Some problems are not very clearly or carefully stated; see for instance, p. 8 (20, 21), p. 124 (43).

The book is very neatly printed, although misprints are not infrequent.

J. D. TAMARKIN

A Treatise on the Analytical Dynamics of Particles and Rigid Bodies; with an Introduction to the Problem of Three Bodies. By E. T. Whittaker. Third Edition. Cambridge, University Press, 1927. xiv+456 pp.

The first edition of this book, which was published in 1904, received an extensive review by E. B. Wilson in this Bulletin (vol. 12 (1906), pp. 451–458). The second edition, which appeared in 1917, (see this Bulletin, vol. 26 (1920), p. 183) differed from the first mainly in the matter of references to or brief outlines of more recent researches.

In the third edition the first fourteen chapters, with some corrections and additional references, have been reproduced photographically from the second edition. Chapter XV on the general theory of orbits and Chapter XVI on integration by series have been completely rewritten in order to present the subject as it has been developed by the researches of the last eleven years. As illustrations of periodic and asymptotic orbits the author has treated the paths which small particles describe in the gravitational field due to a single attracting mass when the newtonian law of gravitation is replaced by the laws belonging to the general relativity theory. There is also a brief account of Synge's geometry of dynamics, in which dynamical problems are treated by aid of the tensor analysis. References only are given to the work of Birkhoff on the classification of the various types of motion of a dynamical system with two degrees of freedom.

Wilson's excellent account of the characteristics of the first edition makes it unnecessary to comment in more detail upon the present edition.

W. R. LONGLEY

Höhere Algebra. By Helmut Hasse. Volume 2: *Gleichungen höheren Grades.* (Sammlung Göschen.) Berlin, de Gruyter. 1927, 160 pp.

This second part of Hasse's algebra is still more interesting than the first. The contents are apparently the same as in most textbooks on higher algebra: the theory of general algebraic equations, Galois group theory, leading up to the condition for a solution by radicals.

This algebra is however no textbook in the common sense of the word, but a modern scientific review of the fundamentals of algebra, in particular the Galois fields, wherein the last traces of the theory of functions in algebra are eliminated.

The definition of the abstract rings and fields and the fundamental properties of them were already given in vol. I. A feature of the present volume is the introduction of the ideas of Steinitz developed in his famous paper *Algebraische Theorie der Körper* (Journal für Mathematik, vol. 137). Steinitz's classification of the abstract fields, fields of characteristic 0 and fields of characteristic p, is introduced: to the last class belong for instance all finite fields, and all fields having a finite subfield. All fields containing the rational field have the characteristic 0. By construction of successive adjunction fields a Galois field is obtained, and the main theorems on the connection between groups and equations are proved for all perfect (vollkommen) fields, that is, all fields with characteristic 0 and fieds with characteristic p having certain properties.

It is difficult to give a satisfactory account of this excellent little book in a few lines, but it can be warmly recommended to *all* mathematicians interested in algebra.

OYSTEIN ORE

Compléments de Géométrie Moderne. By Charles Michel. Paris, Vuibert, 1926. 317 pp. Price (unbound) 35 frs.

The name *Modern Geometry* is used by different writers to cover a vast and varied body of doctrine, from, say, the Simson line and up to the Lie theory, or anything else. What may be meant by "compléments" is still less certain. The preface of the book might be expected to throw some light upon the vague and unassuming title, but the book has no preface. Thus reduced to use the publisher's announcement as a substitute, one soon becomes convinced that a more ambitious caption would come much nearer to be descriptive of the contents of the work. For the author presupposes on the part of the reader a familiarity with the more elementary projective properties of conics, quadrics, and algebraic curves and surfaces in general, with the use of imaginaries in geometry, etc. In other words, he expects his reader to have mastered the advanced parts of a French *Traité de Géométrie*, like Hadamard's, or Rouché et Comberousse's, and in addition a book like E. Duporcq's *Premiers Principes de Géométrie Moderne.*

With this much as a background the author treats pencils and nets of conics and quadrics, the cubic curve both in the plane and in space, Steiner's surfaces. The method of presentation throughout the book is almost exclusively synthetic, except for the first chapter, where the author

accords "au calcul sa part inévitable." That the author thus expedites matters is quite certain, but to say that the analytical attack and the implied principle of continuity are unavoidable is to forget the brilliant and fruitful labors of von Staudt and his successors.

The author has made a very judicious choice in the vast material available and has produced a book which duplicates no other single book in the existing literature. The book is written in the good traditional style of French clarity and is free of excessive condensation. The reader is further helped by the figures which appear when and where they are most needed. Bibliographical references are few in number, there is no index, and the table of contents is brief and summary.

We have quite a few books which may be used as a basis for a first course in synthetic projective geometry, particularly in English. The situation changes radically when it comes to a second course in this subject. M. Michel's book may be used to great advantage for such a purpose in our American schools, and the set of nearly one hundred exercises at the end of the book may be quite helpful in this connection.

NATHAN ALTSHILLER-COURT

Linienspektren und periodisches System der Elemente. By Friedrich Hund. Volume IV of the series Struktur der Materie, edited by M. Born and J. Franck. Berlin, Springer, 1927. vi+221 pp.

In this book a very successful attempt is made to give a recapitulation and a uniform, systematic presentation of the results obtained in the unravelling and classification of line spectra from the pioneer work of Bohr through the investigations by Catalán, Russell, Saunders, Pauli, Heisenberg, Schrödinger, and others. In order to present only one kind of problem to the reader at a time the author has intentionally made very little explicit use of the new quantum mechanics. Nevertheless, he makes a special point of calling attention to the difficulties and limitations inherent in the correspondence principle. Hence, from different points of view the book contains material which should appeal to the astronomer and the chemist as well as to the theoretical physicist and the spectroscopist.

Not only is the presentation relatively non-mathematical, in the sense that advanced analysis has been replaced by a vectorial algorithm devised by the author, but it is also unusually clear and logical. By commencing with the rough and then proceeding to the finer properties of spectra, the atomic model so-called is refined step by step until it becomes possible to give at least a qualitative account of all the details of the most complex spectra. By no means the least valuable feature of the book consists in the numerous, excellent tables which constitute the best and most extensive classification of spectral terms and relationships extant. It is thus evident that the text is an unusually valuable contribution to the field of line spectra.

H. S. UHLER

Interpolation. By J. F. Steffensen. Baltimore, Williams and Wilkins, 1927. 248 pp.

Numerische Infinitesimalrechnung. By Martin Lindow. Berlin and Bonn, Ferd. Dummlers, 1928. 176 pp.

Professor Steffensen's book is the outgrowth of the lectures which the author has given to actuarial students at the University of Copenhagen and is, with a few additions and simplifications, a translation of the Danish edition published in 1925. The book is intended as a text for students in American colleges and requires as mathematical equipment only an elementary knowledge of the differential and integral calculus. In a few places where the gamma function has been used the paragraphs have been printed in smaller type and may be omitted without breaking the continuity of the text.

The topics covered are (1) the general theory of interpolation and extrapolation including the standard formulas usually associated with the names of Newton, Gauss, Stirling, Bessel, and others; (2) numerical differentiation; (3) numerical integration; (4) numerical solution of differential equations.

Professor Steffensen's treatment is more rigorous than is usual in books on interpolation. This is important not merely from the point of view of the pure mathematician but also because of the increased number of formulas with workable remainder terms. It should not be supposed, however, that this adds to the difficulty of reading the text. The style is clear and, after the meaning of the symbols has been mastered, the book should prove very valuable to the increasing number of Americans who require some knowledge of this field of mathematics. The formulas and methods are illustrated by simple numerical examples, but the value of the book for class room use would be increased if it contained some problems to be solved by the student.

Dr. Lindow's book covers essentially the same ground as the preceding one. In the section on the numerical integration of differential equations he has made applications to the problem of the simple pendulum and the restricted problem of three bodies. There are no problems for the student but this book is not intended for class use in American colleges. A useful feature for the computer is the inclusion of seventeen tables giving the numerical coefficients for the expansions required in the various standard formulas.

W. R. Longley

NOTES

The July, 1928, number of the Transactions of this Society (volume 30, No. 3) contains the following papers: *Transversality in space of three dimensions*, by E. Kasner; *On integral equations with discontinuous kernels*, by J. D. Tamarkin and R. E. Langer; *On approximation to an arbitrary function of a complex variable by polynomials*, by J. L. Walsh; *Transformations of nets*, by V. G. Grove; *Allgemeine Eigenschaften der Cantorschen Kohärenzen*, by M. Zarycki; *On the irregular cases of the linear ordinary difference equation*, by C. R. Adams; *On the convergence of quadrature formulas related to an infinite interval*, by J. V. Uspensky; *Second-order linear systems with summable coefficients*, by J. H. Sturdivant; *Concerning the arc curves and basic sets of a continuous curve*, by W. L. Ayres; *A solution of the matric equation* $P(X)=A$, by W. E. Roth; *Concerning the cut points of continua*, by G. T. Whyburn; *Some theorems on the connection between ideals and group of a Galois field*, by O. Ore.

The July, 1928, number of the American Journal of Mathematics (volume 50, No. 3) contains: *On hyperelliptic θ-functions with rational characteristics*, by O. Zariski; *Certain perfect groups generated by two operators of orders two and three*, by H. R. Brahana; *On triadic Cremona nets of of plane curves*, by F. Farnum; *Number relations between types of extremals joining a pair of points*, by D. E. Richmond; *An intrinsic treatment of Poisson's integral*, by F. W. Perkins; *On the invariant combinants of two binary quintics*, by T. W. Moore; *A boundary value problem of ordinary self-adjoint differential equations with singularities*, by M. C. Gray; *Nets of conics in the real domain*, by A. D. Campbell; *Rational tacnodal and oscnodal quartic curves considered as plane sections of quartic surfaces*, by L. T. Moore and J. H. Neelley.

The April, 1928, number of the Annals of Mathematics (series 2, volume 29, No. 2) contains: *The canonical form of a one-parameter group*, by P. Franklin; *A class of real quadratic forms in infinitely many variables*, by F. H. Murray; *The general geometry of paths*, by J. Douglas; *On a geometrical theory of continuous groups.* II. *Euclidean and hyperbolic groups of three-dimensional space*, by B. de Kerékjártó; *A method of numerical solution of the problem of Plateau*, by J. Douglas; *Minimizing two types of definite integral*, by P. R. Rider; *On two types of plane rational curve*, by H. Hilton; *A correspondence between matrices and quadratic ideals*, by C. C. MacDuffee; *A class of functional equations*, by E. Hille; *Definitions of abstract groups*, by G. A. Miller; *Concerning a set of metrical hypotheses for geometry*, by J. L. Dorroh; *Closed point sets on a manifold*, by S. Lefschetz.

A new Bulletin of the National Research Council series has recently appeared, entitled *Selected topics in algebraic geometry*. It contains chapters by A. B. Coble, A. Emch, S. Lefschetz, F. R. Sharpe, C. H. Sisam, and V. Snyder.

The Open Court Publishing Company has just issued, through the University of Chicago Press, the first volume, *Notations in Elementary Mathematics*, of a work entitled *A History of Mathematical Notations*, by Professor F. Cajori, of the University of California.

By the will of Lieutenant Colonel A. J. C. Cunningham, who died February 8, 1928, the London Mathematical Society will receive his library, and also £1,000 for the improvement of methods of factorization of large numbers, and £2,000 for the publication of Colonel Cunningham's works. One twelfth of the residuary estate also will go to the London Mathematical Society, and one twelfth to the mathematical subsection of the British Association, for preparing new mathematical tables in the theory of numbers.

Professor J. W. Lasley, of the University of North Carolina, has been elected chairman of the mathematics section of the North Carolina Academy of Science, and Professor W. W. Elliott, of Duke University, secretary of that section.

The Albert medal of the Royal Society of Arts has been awarded to Sir Ernest Rutherford, for his pioneer researches in the structure of matter.

Dr. Henri Deslandres, director of the observatory of Paris, has received the honorary degree of doctor of mathematics and physics from the University of Leyden.

Professor Emile Picard, of the University of Paris, celebrated, on May 6, 1928, his fiftieth scientific anniversary.

Cambridge University has conferred an honorary doctorate on Professor A. Einstein.

Professors David Hilbert, Paul Langevin, and Ludwig Prandtl have been elected foreign members of the Royal Society of London.

Dr. J. H. Jeans, secretary of the Royal Society, has been knighted.

Dr. Cristóbal de Losada y Puga, of Lima, Peru, a member of this Society, has been elected a corresponding member of the Royal Academy of Sciences of Madrid.

Sir Ernest Rutherford has been elected an associate of the Royal Academy of Belgium.

Williams College has conferred the honorary degree of doctor of science on Professor W. A. Bratton, of Whitman College.

Columbia University has conferred an honorary doctorate on President W. W. Campbell, of the University of California, director of the Lick Observatory.

Professor Joaquin Gallo, director of the National Astronomical Observatory of Mexico, has received the honorary degree of doctor of science from Northwestern University.

The University of Indiana has conferred an honorary degree on Professor J. A. Miller, of Swarthmore College.

Princeton University has conferred the honorary degree of doctor of science on Professor R. A. Millikan.

The chair of physics of Bryn Mawr College has been named in honor of Marion Reilly; for partial support of it, a fund to be known as the Marion Reilly Memorial is being created by a committee of her friends and classmates, under the chairmanship of Mrs. Monroe Buckley.

Professor Karl Pearson has been elected a foreign honorary member of the American Academy of Arts and Sciences.

Professor U. Amaldi, of the Higher School of Architecture at Rome, has been elected a corresponding member of the Reale Accademia dei Lincei.

The R. Accademia delle Scienze of Turin has elected the following corresponding members: E. Fermi, of the University of Rome; A. Signorini, of the University of Naples; L. Tonelli, of the University of Bologna; G. Vitali, of the University of Padua. Professor Vitali has been named corresponding member of the R. Istituto Veneto; he has also been awarded the prize for mathematics for the year 1927 by the Italian Society of Sciences.

Proferror W. B. Ford, of the University of Michigan, has been granted leave of absence for the current academic year and will travel in Europe. Representing the Carnegie Endowment for International Peace, he will lecture at the Universities of Leiden and Utrecht in Holland, the University of Brussels in Belgium, the Universities of Lille and Grenoble in France, and the University of Pisa in Italy.

Professor M. I. Pupin has received the honorary degree of doctor of laws from Middlebury College. He has also recently been elected an honorary member of the American Institute of Electrical Engineers.

Professor D. E. Smith, retired, of Columbia University, has returned from Europe, where he has served as a visiting Carnegie professor of international relations. Representing the Carnegie Endowment for International Peace, he lectured at University College, London, and at the Universities of Montpellier and Toulouse.

Associate Professor Ludwig Berwald, of the German University of Prague, has been appointed to a professorship at the German Technical School at Prague. Dr. Paul Funk has been promoted to a professorship of mathematics at the same technical school.

At the University of Göttingen, Professor H. Bohr, of the Copenhagen Technical School, has been appointed to a professorship of mathematics; Dr. Hans Lipps and Dr. Max Schuler have been promoted to associate professorships.

Dr. Alwin Walther has been appointed professor of mathematics at the Darmstadt Technical School.

Assistant Professor C. R. Adams, of Brown University, has been promoted to an associate professorship of mathematics.

Dr. Constance R. Ballantine has been appointed associate in mathematics at the University of Washington.

Assistant Professor I. A. Barnett, of the University of Cincinnati, has been promoted to an associate professorship of mathematics.

Mr. C. F. Bowles has been promoted to an assistant professorship of mathematics at the South Dakota State School of Mines.

Associate Professor J. A. Bullard, of the United States Naval Academy, has been appointed to a professorship of mathematics at the University of Vermont.

Assistant Professor W. B. Campbell, of Colgate University, has been appointed by the Baptist Mission Board lecturer in mathematics at Judson College, Rangoon, Burma.

Dr. Elizabeth Carlson has been promoted to an assistant professorship at the University of Minnesota.

Dr. M. G. Carman has been appointed head of the department of mathematics at Murray State Teachers College, Murray, Ky.

Associate Professor A. F. Carpenter, of the University of Washington, has been promoted to a professorship of mathematics.

Dr. W. F. Cheney has been promoted to an assistant professorship of mathematics at Tufts College.

Assistant Professor Lennie P. Copeland has been promoted to an associate professorship at Wellesley College.

Assistant Professor C. H. Currier has been promoted to an associate professorship of mathematics at Brown University.

Mr. L. J. Deck has been promoted to a professorship of mathematics at Muhlenberg College.

Assistant Professor R. F. Deimel, of Stevens Institute of Technology, has been promoted to an associate professorship of mechanics.

Dr. R. D. Doner has been appointed associate professor of mathematics at the Alabama Polytechnic Institute.

Assistant Professor W. E. Edington has been promoted to an associate professorship of mathematics at Purdue University.

Assistant Professor H. A. Fales, of Columbia University, has been promoted to an associate professorship of chemistry.

Assistant Professor I. H. Fenn, of the Brooklyn Polytechnic Institute, has been promoted to an associate professorship of mathematics.

Assistant Professor G. H. Graves has been promoted to an associate professorship of mathematics at Purdue University.

Dr. D. C. Harkin has been appointed associate professor at the Alabama Polytechnic Institute.

Assistant Professor D. A. Hatch, of Lafayette College, has been promoted to an associate professorship of mathematics.

Dr. C. E. Horne has been appointed professor of mathematics at the University of Porto Rico.

Assistant Professor Daniel Hull has been promoted to be full professor and head of the department of physics at the University of Notre Dame.

Dr. Vern James has been appointed head of the department of mathematics at Menlo Junior College.

Mr. F. C. Jonah has been appointed assistant professor of mathematics at Adelbert College.

Dr. E. G. Keller has been appointed assistant professor of mathematics at the University of Texas.

Assistant Professor Claribel Kendall, of the University of Colorado, has been promoted to an associate professorship of mathematics.

Associate Professor F. C. Kent has been promoted to a professorship of mathematics at the Oregon State Agricultural College.

Associate Professor E. P. Lane has been promoted to a professorship of mathematics at the University of Chicago.

Mr. H. I. Lane has been appointed assistant professor of mathematics at the University of South Dakota.

Dr. C. G. Latimer has been appointed professor of mathematics at the University of Kentucky.

Dr. E. E. Libman, of the University of Illinois, has been promoted to an assistant professorship of physics.

Dr. R. B. Lindsay has been promoted to an assistant professorship of physics at Yale University.

Dr. R. G. Lubben has been appointed adjunct professor of mathematics at the University of Texas.

Assistant Professor G. F. McEwen has been promoted to a full professorship at the Scripps Institution of Oceanography.

Dr. Elsie J. McFarland has been promoted to an assistant professorship of mathematics at the University of Oklahoma.

Dr. A. J. Maria has been appointed assistant professor of mathematics at the University of Illinois.

Assistant Professor R. W. Marriott has been promoted to a professorship of mathematics and astronomy at Swarthmore College.

Dr. Florence M. Mears has been appointed acting assistant professor of mathematics at Pennsylvania State College.

Professor J. M. Melchiors has been appointed professor of physics at Loyola University, Matoon, Wis.

Dr. G. M. Merriman has been appointed assistant professor of mathematics at Grinnel College.

Associate Professor J. C. Morehead has been promoted to a professorship of graphics in the department of architecture of the Carnegie Institute of Technology.

Associate Professor F. R. Morris has been promoted to be professor and head of the department of mathematics and engineering at Fresno State College.

Assistant Professor D. S. Morse has been promoted to an associate professorship of mathematics at Union College.

Assistant Professor Marston Morse has been promoted to an associate professorship of mathematics at Harvard University.

Associate Professor F. D. Murnaghan, of Johns Hopkins University, has been promoted to a professorship of applied mathematics.

Associate Professor J. R. Musselman, of Johns Hopkins, has been appointed professor of mathematics in the Woman's College of Western Reserve University.

Assistant Professor J. A. Northcott has been promoted to an associate professorship of mathematics at Columbia University.

Dr. H. P. Pettit has been appointed professor of mathematics at Marquette University.

Mr. H. S. Pollard has been appointed assistant professor of mathematics at Miami University.

Assistant Professor R. G. Putnam, of New York University, has been promoted to an associate professorship of mathematics.

Associate Professor L. J. Reed, of Johns Hopkins, has been promoted to a professorship of biometry and vital statistics.

Professor P. R. Rider, of Washington University, has been appointed Sterling Research Fellow at Yale for the year 1928–29.

Associate Professor H. M. Robert has been promoted to a professorship at the United States Naval Academy.

Assistant Professor H. P. Robertson, of the California Institute of Technology, has been appointed visiting assistant professor of mathematical physics at Princeton University for the year 1928–29.

Dr. C. F. Roos has been appointed assistant professor of mathematics at Cornell University.

Dr. W. E. Roth has been appointed assistant professor at the University of Wisconsin, in the university extension division at Milwaukee.

Dr. C. A. Rupp has been appointed assistant professor of mathematics at Pennsylvania State College.

Dr. Hazel E. Schoonmaker has been appointed assistant professor of mathematics at the New Jersey College for Women.

Assistant Professor H. A. Simmons has been promoted to an associate professorship at Northwestern University.

Assistant Professor Herman L. Smith has been promoted to an associate professorship of mathematics at Louisiana State University.

Mr. A. F. C. Stevenson has been promoted to an assistant professorship of mathematics at the University of Toronto.

Dr. M. H. Stone has been promoted to an assistant professorship of mathematics at Harvard.

Dr. Marian M. Torrey has been promoted to an assistant professorship of mathematics at Goucher College.

Dr. P. S. Wagner has been appointed professor of mathematics at Lebanon Valley College, Annville, Pa.

Dr. L. E. Ward has been promoted to an assistant professorship of mathematics at the University of Iowa.

Assistant Professor Warren Weaver, of the University of Wisconsin, has been promoted to a professorship of mathematics.

Dr. G. T. Whyburn has been promoted to an adjunct professorship of mathematics at the University of Texas.

Dr. Frederick Wood has been appointed professor of mathematics at Hamline University.

Miss Frances M. Wright has been appointed to an assistant professorship at Elmira College.

The following appointments to instructorships in mathematics are announced:

University of Arkansas, Mr. D. P. Richardson;

Brooklyn Polytechnic Institute, Mr. D. E. Whitford;
Bryn Mawr College, Mr. H. H. Pixley;
Cornell University, Mr. L. J. Paradiso;
Hunter College, Miss R. L. Anderson;
University of Illinois, Dr. Bessie I. Miller;
Johns Hopkins University, Dr. L. M. Blumenthal;
Lafayette College, Mr. J. H. Fithian;
Lehigh University, Dr. H. B. Hammatt, Messrs. Keeler and Hugh Stanley, Dr. W. J. Trjitzinsky, Mr. A. N. Van Arnam;
University of Michigan, Mr. F. S. Beale;
Oregon State Agricultural College, Miss Melva K. Thompson;
University of Rochester, Miss Rose A. Whelan;
Sweet Briar College, Miss Julia W. Bower;
Williams College, Mr. H. L. Dorwart;
Winthrop College, Miss Alice A. Grant;
University of Wisconsin, Mr. J. I. Vass.

Professor Luigi Bianchi, director of the Scuola Normale Superiore at Pisa, died June 6, 1928, at the age of seventy-two.

Professor Arthur Schoenflies, of the University of Frankfurt on Main, died May 27, 1928, at the age of seventy-five.

Professor Otto Staude, of the University of Rostock, is dead.

Professor Heinrich Weber, formerly professor of physics at the Braunschweig Technical School, is dead.

Professor R. E. Allardice, of Stanford University, died May 29, 1928, at the age of sixty-six. Professor Allardice had been a member of the American Mathematical Society since 1901.

Professor L. W. Dowling, of the University of Wisconsin, died on September 16, 1928, at the age of sixty-one. He had been a member of this Society since 1897.

Mr. J. H. Woodward, consulting actuary, of New York City, died May 15, 1928. Mr. Woodward was a member of the American Mathematical Society.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

APPELL (P.). Le problème géométrique des déblais et remblais. (Mémorial des Sciences Mathématiques, No. 27.) Paris, Gauthier-Villars, 1928.

BAUER (G.). See BIEBERBACH (O.).

BIEBERBACH (O.). Vorlesungen über Algebra. Unter Benutzung der dritten Auflage des gleichnamigen Werkes von Gustav Bauer in vierter vermehrter Auflage dargestellt. Leipzig, Teubner, 1928. 10+334 pp.

BIENIAS (M.). Untersuchung über die kollinearen Erzeugungen der Kummerschen Fläche. Hildesheim, Verlag Borgmeyer, 1928. 39 pp.

BORGER (R. L.). Analytic geometry. New York, McGraw-Hill, 1928. 334 pp.

BRASCH (F. E.), editor. Sir Isaac Newton, 1727–1927, a bicentenary evaluation of his work. Prepared under the auspices of the History of Science Society and cooperating societies. Introduction by D. E. Smith. Baltimore, Williams and Wilkins, 1928. 240 pp.

BURNSIDE (W.). Theory of probability. Edited, with a memoir, by A. R. Forsyth. Cambridge, University Press, 1928. 30+106 pp.

CASTELNUOVO (G.). Calcolo delle probabilità. Volume II. Bologna, 1928. 8+243 pp.

COTTON (E.). Approximations successives et équations différentielles. (Mémorial des Sciences Mathématiques, No. 28.) Paris, Gauthier-Villars, 1928.

FLADT (K.), herausgegeben von. Quellenhefte zur Elementarmathematik. Band 1: Elementargeometrie, von K. Fladt. Leipzig, Teubner, 1928. 73 pp.

FORSYTH (A. R.). See BURNSIDE (W.).

GAMBIER (B.). Déformation des surfaces étudiées du point de vue infinitésimal. (Mémorial des Sciences Mathématiques, No. 26.) Paris, Gauthier-Villars, 1928.

GUICHARD (C.). Les courbes de l'espace à n dimensions. (Mémorial des Sciences Mathématiques, No. 29.) Paris, Gauthier-Villars, 1928.

HILB (E.). See LAGALLY (M.).

KLIEM (F.). Apollonius. Berlin, Salle, 1927. 75 pp.

KOPEČNÝ (J.). Über die Gleichung $x^n+y^n=z^n$. Teil 1. Pressburg, Wigand, 1928. 16 pp.

LAGALLY (M.). Vorlesungen über Vektor-Rechnung. (Mathematik und ihre Anwendungen, herausgegeben von E. Hilb, Band 2.) Leipzig, Akademische Verlagsgesellschaft, 1928. 18+358 pp.

LEVI-CIVITA (T.). Der absolute Differentialkalkül und seine Anwendungen in Geometrie und Physik. (Die Grundlehren der mathematischen Wissenschaften, Band 28.) Berlin, Springer, 1928. 11+310 pp.

LOBATSCHEVSKY CELEBRATION. Ad annum MCMXXVI centesimum a geometra kazaniensi N. J. Lobacewski noneuklideae geometriae systematis inventi concelebrandum. Kazan, Physico-Mathematical Society, 1927. 112 pp.

——In memorian N. I. Lobatschevskii, volume II: Collections des mémoires presentés par les savants de divers pays à la Société Physico-Mathématique de Kazan à l'occasion de la célébration du centenaire de la découverte de la géométrie non-euclidienne. Kazan, 1927.

MALET (H.). Exposé élémentaire du calcul vectoriel et de quelques applications. Avec une préface de Maurice d'Ocagne. Paris, Gauthier-Villars, 1928. 72 pp.

MALSCH (F.). Zahl und Raum. Band I-VIII. Leipzig, Quelle und Meyer, 1927.

NEWTON BICENTENARY. See BRASCH (F.E.).

D'OCAGNE (M.). See MALET (H.).

POPP (K.). Die Bedeutung der Mathematik und Astronomie für die Gegenwartskultur und ihre Entwicklung im Unterricht der höheren Schule. München, Oldenbourg, 1927. 61 pp.

SMITH (D. E.). See BRASCH (F. E.).

SYMON (A.). The new geometry. Part 3: Circles. Glasgow, Robert Gibson, 1928.

WEIDA (F. M.). The logarithmic slide-rule. A pamphlet to simplify instruction. New York, Macmillan, 1927. 7 pp.

WENZEL (A.). Galilei. Berlin, Salle, 1927. 72 pp.

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ABBOTT (C. G.). See MITCHELL (S. A.).

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BENNY (L. B.). Mathematics for students of technology. A senior course. London, Oxford University Press, 1927. 28+451 pp.

BERLAGE (H. P.). Versuch einer Entwicklungsgeschichte der Planeten. Leipzig, Akademische Verlagsgesellschaft, 1927. 68 pp.

BESSON (H.). See HOPPE (E.).

BOEGEHOLD (H.). Geometrische Optik. Berlin, Gebrüder Borntraeger, 1927. 375 pp.

BOLLNOW (O. F.). See EDDINGTON (A. S.).

BRAGG (W. H.). The structure of an organic crystal. London, Longmans, 1928. 32 pp.

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CASPARI (W. A.). The structure and properties of matter. London, Benn, 1928. 78 pp.

CHAZY (J.). La théorie de la relativité et de la mécanique celeste. Tome I. Paris, Gauthier-Villars, 1928.

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DINGLER (H.). Das Experiment, sein Wesen und seine Geschichte. München, Reinhardt, 1928. 259 pp.

DRUMAUX (P.). La théorie des quanta. Paris, Gauthier-Villars, 1927. 152 pp.

DUVAL (A. B.) et HEBRARD (L.). Traité pratique de navigation aérienne. 2e édition. Paris, Gauthier-Villars, 1928. 196 pp.

DYSON (F.). Principles of mechanism. London, Oxford University Press, 1928. 7+296 pp.

EDDINGTON (A. S.). Sterne und Atome. Mit Erganzungen des Autors ins Deutsche übertragen von O. F. Bollnow. Berlin, Springer, 1928. 5+124 pp.

EICHENWALD (A.). Vorlesungen über Elektrizität. Berlin, Springer, 1928. 8+664 pp.

FABRY (C.). Eléments de thermodynamique. Paris, Armand Colin, 1928. 216 pp.

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FÖPPL (A.) und FÖPPL (L.). Drang und Zwang. München, Oldenbourg, 1928.

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GRAETZ (L.). Handbuch der Elektrizität und des Magnetismus. Band V, Lieferung 2. Leipzig, Barth, 1928. 357 pp.

GRAICHEN (E.). Theorie der Wechselströme. Altenberg, Fischer, 1927. 8+64 pp.

GROSSMANN (M.). Darstellende Geometrie für Maschineningenieure. Berlin, Springer, 1927. 8+236 pp.

GÜNTHER (J.). (WALTER DE HAAS.) Was ist Magnetismus? Stuttgart, Franck'sche Verlagsbuchhandlung, 1927. 78 pp.

HAAS (A.). Introduction to theoretical physics. 2d edition. Volume 1. Translated by T. Verschoyle. London, Constable, 1928.

DE HAAS (W.). See GUNTHER (H.).

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HOPPE (E.). Histoire de la physique. Traduit de l'allemand par Henri Besson. Paris, Payot, 1928. 671 pp.

JAMESON (J. M.). Elementary practical mechanics. 3d edition. London, Chapman and Hall, 1927.

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KHAN (H. K.). The discovery and formulation of a new theory of the generation and propagation of wireless waves in space. London, Thacker, 1928.

KNICKERBOCKER (W. S.) editor. Classics of modern science. New York, Alfred Knopf, 1927. 14+384 pp.

KOPPEL (I.). Der Bau der Atome und das periodische System. Leipzig, Voss, 1927. 6+174 pp.

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LEGGETT (B. J.). The theory and practice of radiology. A treatise in 4 volumes. Volumes 1–3. London, Chapman and Hall, 1928. 12+238 +11+308+11+550 pp.

LINKER (P. B. A.). Grundlagen der Wechelstromtheorie. Berlin, Stilke, 1928.

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MAHLER (G.). Physikalische Aufgabensammlung. 3te völlig umgearbeitete Auflage besorgt von K. Mahler. (Sammlung Göschen.) Berlin, de Gruyter, 1927. 136 pp.

MAHLER (K.). See MAHLER (G.).

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METZNER (P.). Das Mikroskop. Ein Leitfaden der wissenschaftlichen Mikroskopie. 2te Auflage des gleichnamigen Werkes von A. Zimmermann. Leipzig und Wien, Deuticke, 1928. 509 pp.

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MÜLLER (H. G.). Behandlung von Schwingungsaufgaben mit komplexen Amplituden und mit Vektoren. Leipzig, Hirzel, 1928. 10+129 pp.

MÖLLER (J.) und POUILLET (C. S. M.). Lehrbuch der Physik. 11te Auflage. Band 5, 2te Hälfte: Physik des Kosmos. Herausgegeben von A. Kopff. Braunschweig, Vieweg, 1928. 12+596 pp.

MÜLLER (W.). Mathematische Strömungslehre. Berlin, Springer, 1928. 9+239 pp.

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POMEY (J. B.). Cours d'électricité théorique professé à l'Ecole Professionnelle Supérieure des Postes et Télégraphes. Tome II. Paris, Gauthier-Villars, 1928. 373 pp.

POUILLET (C. S. M.). See MÜLLER (J.).

RADSCHEK (R.). Über Meeresflut und Rotation. Berlin, Radschek, 1928. 80 pp.

REDFIELD (J.). Music. A science and an art. New York, Knopf, 1928. 332 pp.

REEVE (W. D.). See ROE (H. B.).

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ROE (H. B.), SMITH (D. E.), and REEVE (W. D.). Mathematics for agriculture and elementary science. Boston, Ginn, 1928. 6+354 pp.

SAXELBY (F. M.). An introduction to practical mathematics. New edition. London, Longmans, 1927. 8+232 pp.

SAUNDERS (V. T.). Light. London, Murray, 1927. 7+320 pp.

SCHEFFERS (G.). Lehrbuch der darstellende Geometrie. 2te Auflage. 1ter und 2ter Band. Berlin, Springer, 1927. 10+423+8+441 pp.

SCHRAMM (E.). See FORSTMANN (A.).

SCHULZ (H.). Zeichnerische Behandlung der geometrischen Optik. Schweidenitz i. Schl., Verlag Köhn, 1927. 31 pp.

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DE SPARRE (—.). Mouvement des projectiles autour de leur centre de gravité et conditions à remplir pour assurer leur stabilité. Paris, Gauthier-Villars, 1927.

STARK (J.). See WIEN (W.).

—— Atomstruktur und Atombindung. Berlin, Seidel, 1928. 198 pp.

STRÖMGREN (B.). See STRÖMGREN (E.).

STRÖMGREN (E.) und STRÖMGREN (B.). Zweite Sammlung astronomischer Miniaturen. Berlin, Springer, 1927.

THOMSON (J. J.). Beyond the electron. Cambridge, University Press, 1928. 43 pp.

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UNWIN (W. C.) and MELLANBY (A. L.). The elements of machine design. New and revised edition. Part 1. London, Pitman, 1927. 16+531 pp.

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WADDICOR (H.). The principles of electric power transmission by alternating currents. London, Chapman and Hall, 1928. 19+399 pp.

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WIEN (W.) und HARMS (F.). Handbuch der Experimentalphysik. Band 6: Kapilarität und Oberflächenspannung, von G. Bakker. Band 21: Anregung der Spektren, spektroskopische Apparate, von G. Joos und E. von Angerer; Stark-Effect, von J. Stark. Leipzig, Akademische Verlagsgesellschaft, 1928, 1927. 458+562 pp.

WIMPERIS (H. E.) and SMITH (F. E.). Physics in industry. Volume 5. Lecture 10: The relationship of physics to aeronautical science, by H. E. Wimperis; Lecture 11: Physics in navigation, by F. E. Smith. London, Oxford University Press, 1927. 54 pp.

YOUNG (G.) and BAXTER (H. E.). Mechanics of materials. New York, Macmillan, 1927. 8+451 pp.

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A. VÉRONNET, *Figures d'équilibre et cosmogonie.*

TH. DE DONDER, *Théorie des champs gravifiques.*

S. ZAREMBA, *La logique des mathématiques.*

A. BUHL, *Formules Stokiennes.*

G. VALIRON, *Théorie générale des séries de Dirichlet.*

A. SAINTE-LAGUË, *Les réseaux (ou Graphes).*

R. LAGRANGE, *Calcul différentiel absolu.*

A. BLOCH, *Les fonctions holomorphes et méromorphes dans le cercle-unité.*

M. JANET, *Les systèmes d'équations aux dérivées partielles.*

L. GODEAUX, *Les transformations birationnelles du plan.*

G. RÉMOUNDOS, *Extension aux fonctions algébroïdes multiformes du théorème de M. Picard et de ses applications.*

N. E. NÖRLUND, *Sur la "somme" d'une fonction.*

G. DARMOIS, *Les équations de la gravitation einsteinienne.*

B. GAMBIER, *Déformation des surfaces étudiées du point de vue infinitésimal.*

Many others are in press or in preparation.

For all information, address

Gauthier-Villars et Cie., 55, Quai des Grands-Augustins, Paris

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, October 27, 1928.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than October 6.

BERKELEY, CALIFORNIA, October 27, 1928. (The San Francisco Section.)

Abstracts must be in the hands of the Secretary of the Section, Professor B. A. Bernstein, Berkeley, California, not later than October 6. Professor E. T. Bell will give an address on *The theory of algebraic numbers in the light of Kronecker's program.*

CINCINNATI, OHIO, November 30 and December 1, 1928.

Abstracts must be in the hands of PROFESSOR M. H. INGRAHAM, Associate Secretary of the Society, University of Wisconsin, Madison, Wis., not later than November 9. On Friday afternoon, Professor G. D. Birkhoff will give an address on *The mathematical bases of art.*

LAWRENCE, KANSAS, December 1, 1928. (The Southwestern Section.)

Abstracts must be in the hands of the Secretary of the Section, Professor E. B. Stouffer, Lawrence, Kansas, not later than November 10.

NEW YORK CITY, ANNUAL MEETING, December 27-31, 1928.

Abstracts must be in the hands of the Associate Secretary, Arnold Dresden, 501 West 116th St., New York City, not later than November 29. On Friday morning, the Bôcher prize will be awarded, and Professor W. F. Osgood will speak on the life and work of Bôcher. On Friday afternoon, at the usual joint session with other organizations, Professor Dunham Jackson will give his retiring address as vice-president of Section A of the A.A.A.S.; and Professor J. L. Coolidge, at the invitation of the A.M.S. and the M.A.A., will present a paper entitled *The heroic age in geometry.* Later on Friday afternoon, the sixth Josiah Willard Gibbs Lecture will be given by Professor G. H. Hardy of the University of Oxford, on the topic *An introduction to the theory of numbers.* (The meetings on Monday, December 31, will consist of a symposium on *Wave Mechanics,* in joint session with the American Physical Society.)

CHICAGO, December 31, 1928, January 1, 1929.

Abstracts must be in the hands of PROFESSOR M. H. INGRAHAM, Associate Secretary of the Society, University of Wisconsin, Madison, Wis., not later than December 3.

R. G. D. RICHARDSON,
Secretary of the Society.

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, Menasha, Wis., or 501 West 116th Street, New York.

The initiation fees ($5.00) and the annual dues ($6.00) of members of the Society are payable to the Treasurer of the Society, W. B. FITE, 501 West 116th Street, New York.

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